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HIERARCHICAL ASSIGNMENTS: STABILITY AND FAIRNESS

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Hierarchical Assignments : Stability and Fairness

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Abstract

We study a simple model assigning workers to employers, where each pair of a worker and an employer has a potential joint productivity, and the complete information about the market is contained in the matrix of potential productivities.

Under certain conditions that we specify, the market is hierarchical, in the sense that both workers and employers can be ordered according to ability, and the Pareto optimal assignment in terms of maximal total productivity is achieved by matching the top worker with the top employer and so on. In other words, we describe a market situation in which the above simple matching procedure is optimal.

Some further properties of hierarchies are presented. We can state explicit values for the earnings in the worker optimal and employer optimal solutions. We further show that our hierarchy concept is a discrete analogue to the Ricardian differential rent model of Sattinger (1979), and that the latter one can easily be derived from our model.

We discuss the compatibility problems between fairness and stability of earnings and assignments. In particular, two notions of fairness that seem sensible in general fail to be stable in hierarchical markets. First, pairwise sharing the Pareto optimal product using a fixed sharing rule generates

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stable assignments only under a very restrictive assumption. Second, distributing the same amount to all workers preserves stability only in the extreme situation of an equal distribution of ability among workers. A uniform distribution of ability will always generate a smaller overall production than an uneven distribution in Pareto optimum. We argue for another notion of fairness that turns out to be stable: an average between the worker optimal and employer optimal solutions.

The model can be used to illustrate imperfect competition, economic growth and corruption.

1 Introduction

Most of economic science is about relations between aggregates, and the individual is portrayed as the typical consumer, producer, worker, employer, etc. However, one area where ordering of individuals is crucial is welfare economics, where a tool of analysis is obtained from ordering individuals according to their income into an income distribution.

Economic welfare can be said to rest on two fundaments: income should be distributed as evenly as possible (*egalité*), and everyone should have equal opportunity to 'make it'.

Obviously these principles are contradictory. Just one person making a career destroys the static equal income requirement. A compromise between these two would be an income distribution with few, if any, poor people, a large group of people having more or less the same income, and then a fat tail to the right of individuals who have made careers of varying success. But this is very much the profile of many developed countries. A variety of options for living up to one's special abilities may in fact be an explanation to growth, that is, a movement from a mass of almost equally poor, to varying degrees of prosperity, the successful eventually pulling those who remained behind, away from poverty.

In Tinbergen (1959), specialization has a final steady state of equal income, because there would be a perfect matching between supply and demand for the whole spectrum of abilities. Although Gini coefficients have increased over a long time in developed countries, Tinbergen's thesis seems rather utopian. In a recent paper, Lindbeck and Snower (1996) present a theoretical argument for a development towards what they call a holistic economy, and away from what they regard as a Tayloristic division of labour. In the post-industrial society, versatility, and general intelligence are in higher demand than very specialised abilities that require approximately the same amount of natural talent, and this leads to more inequality in income, not less. There is also an old and strong force working against an even distribution of income across individuals. In his classical works, Roy (1950, 1951) presents some evidence that abilities in a person tend to get multiplied, rather than added, in the production process. Even if both were symmetrically distributed, multiplication would result in a skew distribution of output, and hence earnings, cf. also Cramér (1945, p. 220).

Another field of research where ordering of individuals is common is in the economics of the organisation, where little can be explained without the concept of hierarchy. Furthermore, the old problem of assigning plants to locations (Koopmans and Beckmann,1957) has found new adherants in game theorists, studying matching problems between men and women, or workers and employers (Roth and Sotomayor, 1990).

Here we will present a very simple model assigning workers to employers. We assume that both workers and employers can be ordered according to ability. A worker's ability is measured as the amount of work he/she can accomplish that the employer can sell in a market. The employers can in turn be ordered according to the price at which they can sell their workers' products. Their joint accomplishment is the product of the two abilities. All possible assignments of workers to employers form an assignment matrix which can be so organised that its trace signifies the Pareto optimal production. We then present a simple algorithm for obtaining this solution — matching the top worker with the top employer, etc. We also generalise the abilities of the agents by allowing ability vectors. However, we still assume that the tasks to be performed by workers and marketed by employers are homogeneous; we don't study the assignment across different jobs, analysed e.g. by Rosen (1978). For an excellent survey article on assignment models, cf. Sattinger (1993).

This paper is organised as follows. In Section 2 we start by axiomatically defining an $n \times n$ hierarchical matrix. We then show that such a matrix will always imply that abilities can be ordered, and that the Pareto optimal assignment is obtained by matching the top worker to the top employer and so on. Next, we give examples of hierarchies and show some properties of hierarchies. We can state explicit values for the earnings in the worker optimal and employer optimal solutions. We further show that our hierarchy concept is a discrete analogue to the Ricardian differential rent model of Sattinger (1979), and that the latter one can easily be derived from our model.

In Section 3 we look at economic implications of our model, regarding the compatibility between fairness and stability of earnings and assignments. In particular, two notions of fairness that seem sensible fail to be stable in general in hierarchical markets. First, pairwise sharing the Pareto optimal product using a fixed sharing rule generates stable assignments only under a very restrictive assumption. Second, distributing the same amount to all workers preserves stability only in the extreme situation of an equal distribution of ability among workers. A uniform distribution of ability will always generate a smaller overall production than an uneven distribution in Pareto optimum. We argue for another notion of fairness that turns out to be stable: an average between the worker optimal and employer optimal solutions.

The fourth section concludes with an argument for the evolution of hierarchy-like markets, and discusses possible uses of our model. By letting agents improve on their abilities one can model a growing economy, where growth is generated by individuals becoming more productive and by the boost from their careers. We sketch a simple way of introducing imperfect information. With rational agents and free markets there is no room for a discriminating coalition if all agree on a sharing rule. However, this simple model offers a tool for analysing what happens if these assumptions are not satisfied. If the sharing rule is chosen according to a democratic voting, there is a majority that would like an even distribution of earnings between workers. A corrupt economic system can be contrasted against the fair and Pareto optimal assignments.

2 Mathematical results

2.1 The assignment game

We start by recalling the assignment model of Shapley and Shubik (1972). Let $\{p_1, \ldots, p_n\}$ be a set of workers and $\{q_1, \ldots, q_n\}$ a set of employers. A matching of workers p_i to employer q_j is denoted by $p_i \leftrightarrow q_j$. Every pair p_i, q_j has a potential productivity α_{ij}.

The problem is to find an optimal assignment, i.e. the one that yields maximal total productivity. If we have a matched pair $p_i \leftrightarrow q_j$, its productivity α_{ij} is distributed as *payoff* u_i to worker p_i and $v_j = \alpha_{ij} - u_i$ to employer q_j. An outcome is an assignment combined with a payoff. The outcome is *stable* if it is individually rational (no payoffs are negative) and if it contains no "blocking pairs" of agents that are not matched but who both have an incentive to disrupt their current matches in order to join each other instead. It is well-known that stable outcomes must be Pareto optimal, i.e. the underlying assignment must be optimal, and that no side payments will occur. Therefore, if the agents are numbered such that $p_1 \leftrightarrow q_1, \ldots, p_n \leftrightarrow q_n$ is an optimal assignment, the stable outcomes are given by the payoff vectors $\bar{u}$ and $\bar{v}$ that satisfy:

$$u_i \geq 0, \quad v_i \geq 0, \quad u_i + v_i = \alpha_{ii}, \quad u_i + v_j \geq \alpha_{ij} \text{ for all} i, j.$$

2.1.1 Hierarchies

The problem of finding optimal assignments and stable outcomes is relatively complicated. Hence it might be interesting to consider special situations where the solution is easier to come by. We will now discuss a situation that we will call a "hierarchy", in which there is a natural order (rank) of the P-agents and Q-agents, respectively. The optimal assignment can the be found in a trivial way. Moreover, we present simple explicit expressions for stable payoffs, and we will be able to give an intuitively natural definition of a "fair" outcome.

Let us begin by describing a simple but important special case of a hierarchy, where there exists some measure of competence of workers, $g_1 \leq g_2 \leq \ldots \leq g_n$, and of employers, $k_1 \leq k_2 \leq \ldots \leq k_n$, such that the potential productivity of matching worker p_i to employer q_j is given by the product of competences:

$$\alpha_{ij} = g_i k_j.$$

For this case, it is easy to see that the optimal assignment is to match the most competent worker p_n to the most competent employer q_n, worker p_{n-1} to employer q_{n-1} and so on. This would also hold if e.g. the productivity would be given by the Cobb-Douglas formula: $\alpha_{ij} = A g_i^\beta k_j^\gamma$, where A, β, γ are positive parameters with $\beta + \gamma \geq 1$.

Compare this with the case of a general matrix (α_{ij}) of rank r, which can always be factorized (in multiple ways) as the product of an $n \times r$-matrix $(g_{i\ell})$ and an $r \times n$-matrix $(k_{\ell j})$. This can be interpreted as each worker p_i having a competence vector $\bar{g}_i = (g_{i1}, \ldots, g_{ir})$, and each employer q_j having a competence vector $\bar{k}_j = (k_{1j}, \ldots, k_{rj})$, where the joint productivity is the scalar product:

$$\alpha_{ij} = \bar{g}_i \cdot \bar{k}_j.$$

In an attempt to mimic the rank one case above, one might order workers and employers by increasing norm of their competence vectors: $|\bar{g}_1| \leq \ldots \leq |\bar{g}_n|$ and $|\bar{k}_1| \leq \ldots \leq |\bar{k}_n|$. However, in this general case we *cannot* guarantee that the optimal assignment will be obtained by matching agents according to their order.

Our concept of hierarchy describes certain matrices (α_{ij}) such that there is an intrinsically natural ordering of the agents, and such that matching agents with respect to this ordering produces the optimal assignment.

2.1.2 Definition of a hierarchy

The matrix (α_{ij}) of potential productivities is a **hierarchy** if the following two conditions are satisfied for all indices between 1 and n:

1. If $j < k$ then $\alpha_{ij} \leq \alpha_{ik}$ and $\alpha_{ji} \leq \alpha_{ki}$.

2. If $i < k$ and $j < \ell$ then $\alpha_{il} + \alpha_{kj} \leq \alpha_{ij} + \alpha_{k\ell}$.

In a hierarchy there is an ordering $p_1, \ldots, p_n$ of the P-agents, and an ordering $q_1, \ldots, q_n$ of the Q-agents, that qualitatively measure competence. Condition 1 says that every P-agent will be better off cooperating with a high-ranked Q-agent than with a low-ranked one, and vice versa. Condition 2 says that given a local situation of two P-agents and two Q-agents, the optimal assignment of these is for the high-ranked agents to match with each other, and the low-ranked to match with each other.

It is an easy exercise to verify that the hierarchy conditions can equivalently be formulated using only local comparisons:

1. $\alpha_{ij} \leq \alpha_{i,j+1}$ and $\alpha_{ij} \leq \alpha_{i+1,j}$.

2. $\alpha_{i,j+1} + \alpha_{i+1,j} \leq \alpha_{ij} + \alpha_{i+1,j+1}$.

2.1.3 Examples of hierarchies

We shall now present some simple hierarchies (leaving the easy verifications to the reader). The first example is the one mentioned earlier; if there is some measure of competence of workers, $g_1 \leq g_2 \leq \ldots \leq g_n$, and of employers, $k_1 \leq k_2 \leq \ldots \leq k_n$, then the productivity matrix given by $\alpha_{ij} = g_i k_j$ is a hierarchy. Obviously, the same holds if the potential productivity is $\alpha_{ij} = a(g_i)b(k_j)$ for any monotonic increasing functions $a(g)$ and $b(k)$.

A trivial kind of hierarchy is obtained by $\alpha_{ij} = g_i + k_j$. More generally, taking

$$\alpha_{ij} = f(g_i, k_j)$$

we will always get a hierarchy for any twice differentiable function $f(g,k)$ satisfying

$$\frac{\partial f}{\partial g} \geq 0, \quad \frac{\partial f}{\partial k} \geq 0, \quad \frac{\partial^2 f}{\partial g \partial k} \geq 0.$$

We will consider this type of structure in Section 2.2.

Note that differentiability of $f(g, k)$ is not a necessary condition for a hierarchy; e.g. the famous Leontief function $\alpha_{ij} = \min(g_i, k_j)$ produces a hierarchy.

Remark 1: Artificial competences. All the examples given above depend on given competences g_i and k_j. It might be interesting to note that any hierarchy allows us to canonically define an ordering of the agents, and this ordering can be perceived as an artifical measure of the competence of agents.

Remark 2: Hierarchies and factorized productivity matrices. If (α_{ij}) is factorized as $(\bar{g}_i)(\bar{k}_j)^T$, the conditions for being a hierarchy translate to the following set of inequalities of scalar products:

1. $\bar{g}_i \cdot (\bar{k}_{j+1} - \bar{k}_j) \geq 0$ and $(\bar{g}_{i+1} - \bar{g}_i) \cdot \bar{k}_j \geq 0$.
2. $(\bar{g}_{i+1} - \bar{g}_i) \cdot (\bar{k}_{j+1} - \bar{k}_j) \geq 0$.

A sufficient (but by no means necessary) condition for these inequalities to be satisfied is of course that $\bar{g}_{i+1}$ and $\bar{k}_{j+1}$ are componentwise greater than $\bar{g}_i$ and $\bar{k}_j$ respectively.

2.1.4 Properties of hierarchies

Let us now prove the claimed property of hierachies, that the optimal assignment respects the hierarchic order. As it turns out, for this property only the second hierarchy condition is necessary.

Proposition 2.1 *Given a hierarchy, the optimal assignment is that p_i cooperates with q_i for all i between 1 and n, and hence the optimal productivity is $\alpha_{11} + \alpha_{22} + \ldots + \alpha_{nn}$.*

PROOF. To prove this, we must show that no other assignment can give higher total productivity. Suppose there is one. Then there is some such assignment with maximal i, where i is defined as the least index such that p_i is matched to q_ℓ for some $\ell > i$, and hence q_i is matched to p_j for some $j > i$. But by the hierarchy assumption, Condition 2, we have

$$\alpha_{i\ell} + \alpha_{ji} \leq \alpha_{ii} + \alpha_{j\ell}$$

and hence by matching p_i to q_i and p_j to q_l we can change the assignment to one with at least as great total productivity, but with larger i, contradicting the maximality assumption.

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The above result can be restated in mathematical terms as follows: being a hierarchy is a sufficient condition on a matrix for the trace being the largest sum that can be formed by choosing one element from each row and column.

Moreover, we claim that independently of which stable outcome we have chosen, the payoffs will mirror the hierarchy. In other words, the hierarchy is also a payoff hierarchy. Here the first hierarchy condition is crucial.

Proposition 2.2 *In a hierarchy, the payoff vectors $\bar{u}$ and $\bar{v}$ will in every stable outcome satisfy $u_1 \leq u_2 \leq \ldots \leq u_n$ and $v_1 \leq v_2 \leq \ldots \leq v_n$.*

PROOF. Suppose that the payoff vectors $\bar{u}$ and $\bar{v}$ yield a stable outcome (together with the assignment of p_i to q_i for every i between 1 and n). It is known that in stable outcomes there are no side payments, so $u_i + v_i = \alpha_{ii}$ for all i. Also, stability implies $u_i + v_j \geq \alpha_{ij}$ for all i, j. Now suppose that $i > j$. We have

$$u_i + v_j \geq \alpha_{ij} \Rightarrow u_i + \alpha_{jj} - u_j \geq \alpha_{ij} \Rightarrow u_i - u_j \geq \alpha_{ij} - \alpha_{jj},$$

and $\alpha_{ij} - \alpha_{jj}$ is nonnegative by the first hierarchy condition. Hence $u_i \geq u_j$. In a similar way we get $v_i \geq v_j$.

∎

A further advantage with hierarchies is that they let us state explicit expressions for certain stable outcomes. Shapley and Shubik (1972) showed that the set of stable outcomes form a lattice with a unique Q-optimal outcome. Any other stable outcome is worse for every Q-agent. Dually, there is also a unique P-optimal outcome, which is the best achievable by all P-agents among the stable outcomes. We can explicitly compute these optimal outcomes for a hierarchy.

The key observation is the one made in the proof above, that the payoffs must satisfy

$$u_{i+1} - u_i \geq \alpha_{i+1,i} - \alpha_{i,i}. \tag{1}$$

From this it follows that the Q-optimal outcome, which is the worst possible payoff for the P-agents, cannot be worse for them than the payoff $\bar{u}$ obtained by taking $u_1 = 0$ and then choosing the smallest payoff differences according to the inequality above, i.e. taking $u_{i+1} - u_i = \alpha_{i+1,i} - \alpha_{i,i}$ for $i = 1, \ldots, n-1$. We shall see that this outcome is stable; therefore it is the Q-optimal outcome.

Proposition 2.3 *Suppose that (α_{ij}) is a hierarchy. Then the Q-optimal stable outcome is obtained by the payoff vectors $\bar{u}, \bar{v}$ given by*

$$u_i = (\alpha_{i,i-1} - \alpha_{i-1,i-1}) + (\alpha_{i-1,i-2} - \alpha_{i-2,i-2}) + \ldots + (\alpha_{2,1} - \alpha_{1,1})$$

and

$$v_i = \alpha_{i,i} - u_i$$

for all i between 1 and n.

PROOF. By the foregoing argument, we only need to show that the payoffs $\bar{u}$ and $\bar{v}$, described in the proposition, give a stable outcome. In other words, we must show that for all i, j we have $u_i + v_j = u_i + \alpha_{jj} - u_j \geq \alpha_{ij}$. We distinguish between two cases: $i > j$ and $i < j$, the case of $i = j$ being obvious.

To begin with, note that for $i > j > k$ we have the following rule:

$$\alpha_{ij} - \alpha_{jj} + \alpha_{jk} = \alpha_{ij} - (\alpha_{jj} - \alpha_{jk}) \geq \alpha_{ij} - (\alpha_{ij} - \alpha_{ik}) = \alpha_{ik}. \tag{2}$$

Now let us treat the case when $i > j$. Then

$$u_i + \alpha_{jj} - u_j = (\alpha_{i,i-1} - \alpha_{i-1,i-1}) + \ldots + (\alpha_{j+2,j+1} - \alpha_{j+1,j+1}) + \alpha_{j+1,j}.$$

By using (2) repeatedly (we show the last repetition below) we get

$$u_i + v_j \geq \ldots \geq \alpha_{i,j+1} - \alpha_{j+1,j+1} + \alpha_{j+1,j} \geq \alpha_{ij}.$$

The case when $i < j$ works in an analogous way.

∎

For the P-optimal outcome we can reason analogously; the worst possible payoff for the Q-agents cannot be worse than the payoff v obtained by taking $v_1 = 0$ and the smallest possible payoff differences respecting the inequalities $v_{i+1} - v_i \geq \alpha_{i,i+1} - \alpha_{i,i}$ for $i = 1, \ldots, n-1$. By a completely analogous argument to the proof above we get the dual result.

Proposition 2.4 *Suppose that (α_{ij}) is a hierarchy. Then a P-optimal stable outcome is obtained by the payoff vectors given by*

$$v_i = (\alpha_{i-1,i} - \alpha_{i-1,i-1}) + (\alpha_{i-2,i-1} - \alpha_{i-2,i-2}) + \ldots + (\alpha_{1,2} - \alpha_{1,1})$$

and

$$u_i = \alpha_{i,i} - v_i$$

for all i between 1 *and n.*

Moreover, if $(\bar{u}^Q; \bar{v}^Q)$ is the Q-optimal outcome and $(\bar{u}^P; \bar{v}^P)$ is the P-optimal outcome, then any weighted mean of the type

$$(\lambda \bar{u}^Q + (1-\lambda)\bar{u}^P; \lambda \bar{v}^Q + (1-\lambda)\bar{v}^P)$$

is also a stable outcome for any value of the parameter λ between zero and one.

2.2 Relationships between the hierarchy model and Sattinger's differential rents model

Sattinger (1979, 1993) considers an assignment model, called the Ricardian differential rents model, where there is a continuous distribution of workers' skills and a continuous distribution of machine sizes. Given certain assumptions on the potential productivity function, he can compute the (unique) stable matching of workers and machines and the wage function. We will show that Sattinger's results can be derived from ours.

2.2.1 The differential rents model

Each job (in Sattinger's terminology: each machine) is described by a single characteristic, say its size, and each worker is described by a single skill. If g_i is a measure of worker i's skill and k_j is a measure of the size of machine j, we suppose that the potential productivity α_{ij} of matching worker i to machine j is described by $\alpha_{ij} = f(g_i, k_j)$ where $f(g, k)$ is an increasing function of both g and k, and has continuous first and second derivatives, such that the mixed partial derivative is positive:

$$\frac{\partial^2 f}{\partial g \partial k} > 0.$$

Sattinger now lets the number of workers and jobs grow indefinitely so that we get a continuous distribution of skills and machine sizes.

Given this model, Sattinger argues that the assignment in equilibrium of workers to jobs will be strictly top–down, that is, the nth worker, in order of decreasing skill, will be assigned to the nth machine, in order of decreasing size. From the continuous distributions of skills and machine sizes it then follows that the relationship between them can be described by a monotonous function $k(g)$ determining which machine size will be assigned to a worker of skill g.

Then Sattinger proceeds by determining the wage function in equilibrium, i.e. the amount $w(g)$ of the productivity that the employer will pay for a worker of skill g. He shows that the wage function must satisfy

$$w'(g) = \left[\frac{\partial f(g, k)}{\partial g}\right]_{k=k(g)} \tag{3}$$

where the partial derivative is taken treating k as a constant. This equation determines the wage function up to a constant term. The constant term, describing the absolute level of wages, will be determined by the conditions for the last match in order of decreasing skill and machine size, basically by choosing a minimum wage.

2.2.2 Deriving the differential rents model from hierarchies

We shall now see how the differential rents model can be derived from our theory of hierarchies. Recall that a hierarchy is a matrix (α_{ij}), with $i, j = 1, 2, \ldots, n$, of potential productivities satisfying

$$\alpha_{ij} \leq \alpha_{i,j+1}, \qquad \alpha_{ij} \leq \alpha_{i+1,j} \tag{4}$$

and

$$\alpha_{ij} + \alpha_{i+1,j+1} \geq \alpha_{i,j+1} + \alpha_{i+1,j}. \tag{5}$$

We can then introduce an artificial measure of workers' skills and machine sizes by defining $g_i = i/n$ and $k_j = j/n$ respectively — although for our purposes any bounded (uniformly for all n) monotonic functions $g(i)$ and $k(j)$ would do. Further, define the function $f(g, k)$ on a discrete set by $f(g_i, k_j) = \alpha_{ij}$.

With these definitions, and letting n tend to infinity, we obtain a uniform distribution of skills and machine sizes in the unit interval. Consider the function $f(g, k)$ that is now defined on the unit square. In the limit, Eq. (4) says exactly that $f(g, k)$ is increasing in both g and k, while Eq. (5) says exactly that the mixed partial derivative $\frac{\partial^2 f}{\partial g \partial k}$ is positive. Hence, we have obtained the differential rents model as the limit case of hierarchies.

Let us now derive the differential equation (3) for the wage function $w(g)$ given by $w(g_i) = u_i$. For a discrete hierarchy we have already computed the Q-optimal and P-optimal stable outcomes in Proposition 2.3 and Proposition 2.4 respectively. In the Q-optimal outcome we had

$$u_i - u_{i-1} = \alpha_{i,i} - \alpha_{i,i-1}$$

while in the P-optimal outcome we had

$$u_{i+1} - u_i = \alpha_{i+1,i} - \alpha_{i,i}.$$

The function $f(g,k)$ being continuously differentiable means that in the limit the left-hand derivative equals the right-hand derivative:

$$\alpha_{i,i} - \alpha_{i,i-1} = \alpha_{i+1,i} - \alpha_{i,i}.$$

Hence we have that both the Q-optimal and P-optimal wage functions satisfy the differential equation (3), so in fact they coincide (except for a constant term) and thus we have a unique wage function in equilibrium, up to the absolute level of wages.

3 Questions of fairness and stability

Given the total productivity of an assignment, it is tempting to devise a rule for sharing the wealth that would be perceived as fair. It is striking how hard it is to find such a rule that at the same time gives stability. A society could of course outlaw reassignment of blocking pairs, and we shall discuss this option briefly in the concluding section. Here, we will look for maximal fairness among stable outcomes. We shall discuss two main approaches that will in general fail to be stable for hierarchies: (a) equal payoff to all workers, and (b) equal sharing between worker and employer. Finally, we propose a notion of fair sharing in hierarchies that is always stable, namely a linear compromise between the Q-optimal and P-optimal outcomes.

3.1 Equal payoff to all workers

One attempt towards achieving fairness is to give the same payoff to all workers, i.e setting $u_i = u$ for all i. When can such an outcome be stable? As usual, let the ordering of agents be such that the optimal productivity is given by the diagonal sum $\alpha_{11} + \ldots + \alpha_{nn}$, with increasing α_{ii}. Then the employers' payoffs are $v_i = \alpha_{ii} - u$. For this outcome to be stable, we must check two things. First, individual rationality: the conditions $u \geq 0$ and $v_i \geq 0$ for all i can with our assumptions be expressed as

$$0 \leq u \leq \alpha_{11}. \tag{6}$$

Second, absence of blocking pairs: the condition $u + v_j \geq \alpha_{ij}$ for all i, j is, together with the assumption $v_j = \alpha_{jj} - u$, equivalent to

$$\alpha_{jj} \geq \alpha_{ij} \quad \text{for all } i, j. \tag{7}$$

Hence, we have shown that giving the same payoff to all workers results in a stable outcome if and only if the workers' payoff is chosen in the interval $[0, \alpha_{11}]$ and the matrix (α_{ij}) is diagonal-heavy in the sense that in each column the diagonal element is the largest one.

3.1.1 Equal payoff to all workers in a hierarchy

A hierarchy (α_{ij}) can satisfy the extra condition of being diagonal-heavy only in the special case when in every column all elements are equal from the diagonal and downwards. This can be interpreted as follows. If assignments are made for employers $q_n, q_{n-1}, \ldots, q_1$ in turn, then at any point worker p_i assigned to q_i is indistinguishable competencewise to the employer from the workers already assigned, and hence he/she might as well get the same payoff.

An interpretation of this situation is that ability is uniformly distributed among workers. In reality this is hardly the case. And for good reasons. It is easy to show that if there were a total amount of ability to be distributed among workers, and if the relation between abilities and production is multiplicative, cf. Roy (1950, 1951), then a uniform distribution would result in the smallest Pareto optimal total production.

3.2 Equal sharing between worker and employer

Another possibility of defining fairness would be that every pair of a worker and an employer should share their joint productivity equally between them, i.e. $u_i = v_i = \alpha_{ii}/2$. This outcome will not always be stable. In fact, since $\alpha_{ij} \leq u_i + v_j$ must hold for all i, j in a stable outcome, we see that $\alpha_{ij} \leq (\alpha_{ii} + \alpha_{jj})/2$ is a necessary and sufficient condition for the stability of the "fair" outcome. It is possible to construct hierarchies having this property, but it would be an extreme case.

A modified version of this notion of fairness is to let $u_i = \lambda\alpha_{ii}$ and $v_i = (1-\lambda)\alpha_{ii}$, where λ is a universal parameter between zero and one. In this case, the necessary and sufficient condition for stability is

$$\alpha_{ij} \leq \lambda\alpha_{ii} + (1-\lambda)\alpha_{jj} \quad \text{for all } i, j.$$

However, it is not easy to find such a λ. So this method, intuitively appealing as it might be, will not be stable and survive on a free market.

3.3 A stable notion of fairness

The most typical feature of a hierarchy is that a more competent worker will always get a higher income than his/her less competent colleagues in any stable assignment. All definitions of fairness, compatible with stability, must take this into account. Our first definition of fairness, giving the same payoff to all workers, failed for this reason.

But stable assignments do allow for a certain degree of freedom in the distribution of incomes. We have seen in Propositions 2.3 and 2.4 that the best possible payoff worker p_i can get is u_i^P and the worst possible outcome is u_i^Q. The payoff to worker p_i must be somewhere in the interval $[u_i^P, u_i^Q]$. The equally shared productivity $\alpha_{ii}/2$ might possibly lie outside this interval. Our second definition of fairness failed for this reason.

The notion of fairness we propose is for the workers and employers to compromise between their respective optimal outcomes, and hence set $u_i = (u_i^P + u_i^Q)/2$ and $v_i = (v_i^P + v_i^Q)/2$. Explicitly, this "fair outcome" takes the shape:

$$u_i = \frac{\alpha_{i,i} - \sum_{j=i+1}^{n}(\alpha_{j,j-1} - \alpha_{j-1,j})}{2},$$

and

$$v_i = \frac{\alpha_{i,i} + \sum_{j=i+1}^{n}(\alpha_{j,j-1} - \alpha_{j-1,j})}{2}.$$

Of course, one could object and say that it is unreasonable to expect the workers and employers to compromise on equal terms. This kind of considerations can be taken care of by introducing a weighted compromise, with a universal parameter λ between zero and one, so that $u_i = \lambda u_i^P + (1-\lambda)u_i^Q$ and $v_i = \lambda v_i^P + (1-\lambda)v_i^Q$.

As we saw when studying Sattinger's continuous model, when the numbers tend to infinity the P-optimal and Q-optimal solutions will differ by a constant term only, in which case *all* stable solutions will be on the line

$$\lambda(\bar{u}^Q; \bar{v}^Q) + (1-\lambda)(\bar{u}^P; \bar{v}^P).$$

This is a kind of justification for looking for fair outcomes on this line in the discrete case too.

4 Conclusions and Applications

We have discussed a model for assigning workers to jobs, and in particular we have studied a certain kind of market which might be named 'hierarchical', since it induces a linear ordering of workers and employers according to ability. These hierarchies were mathematically convenient objects, in that they allowed us to state explicitly the worker optimal and employer optimal stable distribution of earnings; we then suggested that a fair distribution should be an average between these two extremes. But how applicable is this model? Certainly the idea of a linear ordering of *all* workers in the world is counterfactual, since different kinds of jobs emphasize different skills of workers. But in a particular sector, the idea may not be too farfetched. As noted in the introduction, a Tayloristic division of labor could be something of the past, especially in the more developed segments of the post-industrial society. Under the assumption that the ability to increase one's skill is positively correlated to one's current position, a dynamic model of repeated assignments would eventually lead to a hierarchy-like situation.

Another aspect of letting abilities increase and repeating the assignment game over and over again is that of modelling a growing economy. Production will grow over time, both because abilities increase, but also because new, and more productive matches are formed. Suboptimal assignments, because of imperfect information, or repressive and corrupt regimes will then automatically lead to slower growth.

How could imperfect information be introduced into the model? A very simple way is to define a diagonal matrix, where some probability rule assigns a one or a zero in the diagonal entries. When multipying the ability vectors (or matrices) one just inserts this matrix between the two, thus preventing the assignments corresponding to zero entries. The result could be either unemployment, or a suboptimal assignment.

Finally, we shall return to the question of equal earnings among workers. In Section 3 it was said that this could produce a stable matching in a hierarchy only if the distribution of ability among workers is essentially even. Introducing strong unions and centralism into the model makes equal pay more plausible. Suppose that workers and employers have agreed on how they should share earnings as groups. Then workers collectively decide according to a majority vote between two sharing rules, where one calls for earnings in proportion to ability, while the other suggests equal distribution. Then a majority would vote for equality. This is because the productivity distribution is skewed, so that there are fewer workers losing than gaining. In the long run, this would lead to shirking and would discourage workers' ambition to improve their ability, thus slowing down growth.

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Sammandrag på svenska

Hierarkiska uppdrag: stabilitet och rättvisa

Vi studerar en modell som parvis förenar arbetare och arbetsgivare i uppdrag. Varje par, har en potentiell produktivitet. All information om marknaden finns i matrisen över potentiella produktiviteter.

Under vissa villkor som vi specificerar är marknaden hierarkisk i bemärkelsen att både arbetare och arbetsgivare kan rangordnas enligt förmåga. Den Pareto-optimala hopparningen i uppdrag är den som ger den största sammanlagda produktionen. Den får man genom att kombinera den bäste arbetaren med den bäste arbetsgivaren, osv. ända ner till det par som utgörs av de sämsta av vardera sorten.

Vi anger några egenskaper hos hierarkier. Det finns en arbetaroptimal, respektive en arbetsgivaroptimal fördelning av förtjänsten. Explicita uttryck ges för dessa. Vi visar att vårt hierarkibegrepp är en diskret analog till Sattingers (1979) modell för rikardiansk differentialränta, som lätt kan härledas ur vår modell.

Diskussionen förs vidare till frågan om hur man skall kunna dela *rättvist* på arbetsresultatet, så att delningen och uppdragsstrukturen förblir *stabil*. Det visar sig att två föreställningar om rättvisa vanligen leder till en instabil lösning. En parvis delning av den Pareto-optimala produkten enligt en fast delningskvot leder till stabilitet bara om man gör ett mycket restriktivt antagande. Försöker man dela

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produktionen så att alla arbetare får lika mycket blir lösningen stabil endast om alla arbetare besitter precis samma förmåga. Skulle så faktiskt vara fallet får man den lägsta tänkbara Pareto-optimala totalproduktionen. Vi argumenterar för ett stabilt rättvisebegrepp som innebär att man tar ett medeltal av den arbetar- och den arbetsgivaroptimala lösningen.

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