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EQUILIBRIUM IN AN ELECTRONIC OPEN LIMIT ORDER BOOK

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Equilibrium in an
Electronic Open Limit
Order Book

by

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¹ This paper was formerly and immodestly titled "The Inevitability and Resilience of an Electronic Open Limit Order Book." Comments would be greatly appreciated. I have benefitted from the insights of Matt Spiegel, Subra Subramanyam, and the comments of seminar participants at Baruch, Rutgers, NYU, the Atlanta Fed, University of Michigan, Northwestern, University of Chicago and Ohio State.

ABSTRACT

Under fairly general conditions, the paper derives the equilibrium price schedule determined by the bids and offers in an open limit order book. The analysis shows that 1) the order book will have a small trade spread, and small trades will be profitable while larger ones will not; 2) the electronic exchange provides as much liquidity as possible in extreme situations. An analysis of competing exchanges that are anonymous shows that 1) the electronic exchange does not invite competition in that an entering exchange would expect non-positive trading profits; 2) were a competing exchange to enter anyway, in order for the entering exchange to earn non-negative trading profits, the price schedule from the point of view of investors would look just like the price schedule established in the electronic exchange; and 3) the electronic exchange is uniquely (within a class of anonymous exchanges) immune to competition. The general analysis is illustrated with examples.

This paper provides an analysis of an idealized electronic open limit order book. The focus of the paper is the nature of equilibrium in such a market, and how an open limit order book fares against competition from other methods of exchanging securities. The alternative methods of exchanging securities is limited to those that are anonymous--no trader knows who the counter-party to a trade is. The analysis suggests that an electronic open limit order book mimics competition among anonymous exchanges. Consequently, there is no incentive to set up a competing anonymous exchange given the presence of the limit order book. On the other hand, any other anonymous exchange will invite competition. These conclusions suggest that an electronic open limit order book of the sort considered here has a chance of being a center of significant trading volume. The analysis does not imply that an electronic limit order book will be or should be the only trading institution. It does suggest some of the characteristics that an alternative institution should have to successfully compete with an electronic exchange. The results are obtained in a fairly general environment, and hence would appear to be robust.

The motivation for the paper lies in recent developments in information processing technology and the interest in institutional innovation in the securities industry. It appears that the securities industry is uncertain about future developments in trading institutions. Such systems as INSTINET, the "Wunsch Auction" and electronic trading on the regional exchanges represent different approaches to the use of the information processing technology. The results in this paper are indicative of the direction such developments will take. While the paper does not describe an evolutionary process that will arrive at an open limit order book, the model of investor behavior and competition among exchanges does suggest that the open limit order book is a stable institution and, within the set of economic environments considered, the only stable institution.

The model assumes away a number of frictions and costs that may well be important. The model deals with the architecture of the open limit order book only in general terms, and does not deal with certain technological issues like the computing capacity required and the time required for

computations. In a similar vein, competing exchanges are assumed to be costless to establish. Neither does the model deal with the costs associated with clearing and investor confidence in the clearing process. Harris (1990) is an excellent reference for an analysis of some of these and other issues. Also, see Domowitz (1991). Certain other limitations will be discussed below in the concluding remarks.

There are a number of important antecedents to this work. Trading on private information is an important aspect of the analysis—without it, all of the propositions become trivial. As in Kyle (1985) investors may submit orders of any quantity, but in contrast orders arrive one at a time. This feature recalls Easley and O'Hara (1987) and Glosten (1989). The design of the trading mechanism is, however, different from both of these models (non-discriminatory competitive auction in Easley and O'Hara (1987) and non-discriminatory competitive auction and monopolist specialist in Glosten (1989)). Furthermore, the environment is more general.

The model of the open limit order book and the specification of equilibrium are very similar to the limit order book analysis in Rock (1989). The most important difference is that the model here does not allow a specialist or market maker to disrupt trading against the book. A key feature of the Rock (1989) model is that a market maker can foist a second adverse selection problem on to those providing bids and offers--the book is only hit if the market maker decides to back away (because of order size) from a trade. A second difference is that the quantities traded in the Rock model are exogenous, whereas they are determined endogenously in this paper. This allows an analysis of market breakdown, and is very important for the analysis of competing exchanges.

The discussion in Black (1992) was a major inspiration for this analysis. Indeed, it is possible that the institution considered in Black (1992) will, under certain circumstances be identical to the one considered here. In an earlier version, Black (1991) an institution was developed that used taxes and subsidies to break the equivalence of "net price" and revised expectations in response to trade. This

paper shows that a similar structure of implicit taxes and subsidies can arise in equilibrium via the design of the trading mechanism. The relation between the revised "Black Institution" and the one considered here will be discussed further in the concluding remarks.

It is perhaps useful for the reader to have a guide to the subsequent analysis. The electronic open limit order book is modeled as a publicly visible screen providing, in principle, an infinite number of bids and offers each for a specified quantity. Orders against the book "pick off" the bids or offers in a discriminatory fashion. For example, if each bid or offer is for 100 shares, a transaction of 1000 shares will pick off the ten lowest offers or the ten highest bids at each limit price. Thus, an order of 1000 shares could lead to ten separate transactions at ten distinct prices. The actual transactions are presumed to be the result of rational optimization on the part of risk averse investors, and bids and offers are assumed to reflect this along with the fact that some trades may be motivated by private information. The presence of private information is modelled as an affiliation between the investor's marginal valuation and the future value of a share of stock. The source of bids and offers is a large population of risk neutral "patient traders." The large population and risk neutrality implies a zero expected profit condition on the bids and offers.

After setting up the economic environment and analyzing the trades of investors who trade against the book of limit orders, the paper presents an analysis of the bids and offers that will be provided. In an environment with discrete prices, the actual bids and offers submitted are seen to be related to, respectively, "lower tail" and "upper tail" conditional expectations. This is due to the "discriminatory" nature of the book and the fact that limit orders are picked off in succession. For example, the smallest ask will be hit on any purchase. Thus, the smallest ask must be at least the expected value of the asset conditional on a purchase of any size. The next strictly higher ask (if there is one) must be at least as large as the expected value of the asset conditional on a purchase of size greater than the quantity offered at the first ask price. The possibility of information motivated

trade, as formulated here implies that the schedule of offers is generally upward sloping--it costs more per share to purchase a large number of shares than to purchase a small number of shares. An informational motive for trade also implies that there will be a positive small trade bid ask spread. Furthermore, this spread is bounded below by an amount that is independent of the discreteness of prices. This follows from the observation that the lowest ask is related to an "upper tail" conditional expectation. This will also imply that small trades will be profitable for liquidity suppliers. The zero profit result implies that at least some larger trades will be unprofitable.

The remaining propositions concern comparisons between the electronic exchange and other possible exchanges. The architecture of other possible exchanges is not explicitly considered; rather they are assumed to be characterized by the schedule of prices they will offer. This rules out non-anonymous exchanges which can offer different terms of trade to different individuals. Technical considerations require limiting these exchanges further to the class of exchanges which provide marginal price schedules that are "nice" in that the investor's optimal choice is the unique solution to a first order condition.

While the open limit order book has a small trade spread, it does as well as can be expected at handling extreme adverse selection problems--if no liquidity is supplied by the open limit order book, then every other exchange would expect to lose money by staying open for trade. The reason for this is that the architecture of the open limit order book leads to an averaging of profits across trades.

Since there is a small trade spread, it is possible that even the smallest trades lead to some revision in expectations. This observation is used to show that it is not optimal to split trades into a large number of very small trades. Each small trade has a non-trivial effect on the subsequent terms of trade. Hence, a large number of very small transactions will substantially move the schedule of bids and offers against the transactor.

The next propositions show that the open limit order book is uniquely immune to competing

exchange "cream skimming" of orders when the only way to ascertain "cream" is with trade size--i.e. competing exchanges are anonymous. The key assumption here is that investors can costlessly split their orders among competing exchanges. Thus, in general, if there are several competing exchanges, a single individual's desire to trade will actually result in several separate transactions. Recalling the discriminatory design of the open limit order book, it can be seen that the book already breaks up every desired trade into many smaller transactions (each at the lowest (highest) offers (bids)), and furthermore, the profits from such a break up are competed away. Thus, there is no way to introduce a competing exchange that will get investors to break up their order further and yield a profit. That is, the discriminatory limit order book mimics the competition among exchanges. On the other hand, if some other institution exists which is not competing away the profit from breaking up an order, there is an exchange which can enter and receive an expected profit from an alternative break up of orders.

The subsequent sections defend the above description with a more rigorous analysis. After setting up the environment the above results are derived and discussed further. The general analysis is illustrated at various points with examples. The conclusion identifies and discusses the limitations of the results and points to further analysis.

II. Equilibrium in the Electronic Market

This section of the paper lays out some general characteristics of the electronic open limit order book as conceived here. It is presumed that all potential participants in the market have available an electronic screen which provides a list of all limit orders, buy and sell, that have been entered. The screen is anonymous in the sense that all that is provided are the terms of the limit orders--price and quantity--the identity of the limit order suppliers is not provided. If an individual wishes to add a bid or offer to the market, this can be done costlessly. Furthermore, any bid or offer

may be costlessly retracted at any time except in the middle of the execution of a trade. Execution of a trade against the book occurs in a "discriminatory" fashion. That is, if a trade is large enough to execute against several limit orders at different prices, each limit order transacts at its limit price. For example, if there were two offers at 50 for a thousand shares each, and two offers at 51, each for a thousand shares, a four thousand share purchase would in effect lead to four transactions--two at 50 and two at 51. The marginal price for this four thousand share trade would be 51, while the average price would be 50.5. The total amount paid would be 202,000. The market as conceived here is discriminatory in the same sense that a discriminatory auction discriminates, and since the order book is open the marginal price function must be non-decreasing. It should be noted in passing, that one could also imagine a non-discriminatory electronic limit order book. Analogous to a non-discriminatory auction, a non-discriminatory order book would transact all limit orders at the same price. There are reasons for considering the non-discriminatory book, and these will be discussed below.

Four assumptions will be made restricting the behavior of participants: 1) investors who trade against the book are rational and risk averse in that they choose their trade to maximize a quasi-concave function of their cash and share position; 2) there is the possibility of informed trade in that an investor's marginal valuation is affiliated with the future payoff of the security; 3) there are a large number of risk neutral limit order submitters; 4) in the presence of more than one exchange, investors can costlessly and simultaneously split their order among the exchanges.

The next subsection will describe the behavior of individuals who trade against the book. Following that will be a discussion of the behavior of those who submit limit orders to the book. Those who trade against the book take the terms of trade as given, while those who supply the limit orders take into account the behavior of investors.

The analysis takes place at a point in time. Though some expectations and probabilities will

be written as unconditional, they should be understood to be conditional on all past public information. Similarly conditional probabilities and expectations should be understood to be conditional on the specific argument as well as on all past information. The analysis thus looks at the terms of trade provided conditional on all past public information: the trade made in response to these terms, all past information and possibly some private information; and subsequent revisions in expectations in response to this trade. After the trade, a new public information set is determined--the original public information set plus the trade that occurred. At that point, new terms of trade are determined in the same manner.

II.A Investor Behavior

When the terms of trade are determined, in general, limit order submitters do not know what the next order will be--the next trade is a random variable. The purpose of this subsection is to derive some characteristics of this random variable. Of particular importance is the relation between the random variable and the actual terms of trade. The key assumption is that an investor who trades against the book has determined the trade to maximize something. Furthermore, an assumption will be made to insure that maximization can be described as the unique solution to a first order condition.

The next trader to come to the market may have a number of unobservable characteristics. That these characteristics are unknown before hand is why the next trade is, from the point of view of those providing limit orders, a random variable. The analysis will use the notation, ω , to indicate a vector of unobservable characteristics. Since the trader knows his or her characteristics, the "utility" maximizing trade is determined conditional on this vector of characteristics and the terms of trade. The terms of trade are determined by the list of bids and offers available. The schedule of bids and offers is summarized by the function $R'(q)$. For q positive (an investor purchase), $R'(q)$ is the highest

ask price paid for a purchase of q shares. For q negative (an investor sale), $R'(q)$ is the lowest bid price received for a sale of $-q$ shares. The function $R'(q)$ is precisely the data supplied by the screen and it is the fundamental data describing the terms of trade. The "prime" notation is used to remind the reader that $R'(q)$ is a marginal price. For any q , $R(q)$ is defined to be the (Lebesgue) integral of $R'(\cdot)$ from zero to q . Thus (if all prices are positive), if q is positive $R(q)$ is positive and represents the total amount paid for a purchase of q shares. If a q is negative, $R(q)$ is negative and $-R(q)$ is the amount received for a sale of $-q$ shares. It should be noted that $R'(q)$ may have discontinuities. Thus, while $R(q)$ must be continuous in q it need not be differentiable and hence while $R(q)$ is the integral of $R'(q)$, $R'(q)$ is not necessarily the derivative of $R(q)$.

With this notation, the following assumption regarding investor behavior is offered.

Assumption A1.

An arriving investor with a vector of characteristics, ω , facing a schedule of bids and offers described by the function $R'(\cdot)$, chooses a quantity to trade, q , to maximize $W(-R(q), q; \omega)$. The function $W(c, q; \omega)$ is strictly quasi-concave in (c, q) and strictly increasing in c for all ω . That is, if W_i indicates the first partial derivative of W with respect to the i th argument and W_{ij} indicates the second partial derivative with respect to arguments i and j , then $W_1 > 0$.

$$W_2^2 W_{11} + W_{12} W_{22} - 2W_1 W_2 W_{12} < 0.$$

The first argument of W represents the change in the cash position of the investor as a result of a trade, while the second argument represents the change in the investor's position in the security as a result of a trade. That W is strictly increasing in the first argument means that more cash is preferred to less. Quasi-concavity of W in (c, q) means that in the (c, q) plane, indifference curves are convex to the origin. As the following examples show, it is related to an assumption of risk aversion.

The formulation of investor behavior in assumption 1 encompasses a reasonably wide range of specific behavioral assumptions. Some examples are provided below.

Example--myopic portfolio adjustment

Define $W(c,q;\omega)$ by $W(c,q;\omega) = E[U(Y_T + (\nu+q)X_T + (\phi+c)(1+r_T); Y_T, X_T) | S]$. In this case, the investor chooses q to maximize the expected (possibly state dependent) utility of wealth at time T in the future. The investor has other sources of wealth represented by Y_T , has an initial position, ν , in the security in question and an initial cash position of ϕ . The investor earns a risk free return r_T over the T periods. Furthermore, the investor has a (possibly null) signal about the future random variables. The vector of unobservable characteristics would consist of the specification of the utility function, the time horizon, the joint distribution of Y_T and X_T , the initial cash and security positions, the risk free rate obtained and the nature of and realization of the signal S . Quasi-concavity of W is implied by concavity of U , and $W_1 > 0$ is implied by positive marginal utility of wealth. The formulation is myopic in the sense that the investor ignores future opportunities to trade. An informational motive for trade results from non-null S , while a "liquidity" motive for trade arises from suboptimal ν and ϕ given the random variables Y_T and X_T .

Example--Consumption and Investment

Define $W(c,q;\omega)$ by

$$W(c,q;\omega) = U_0(c^*(c,q,\omega)) + E[U_T(Y_T + (\phi+c-c^*(c,q,\omega))(1+r_T) + (\nu+q)X_T) | S],$$

where $c^*(c,q,\omega)$ is the optimal current consumption given all other characteristics, and other variables are as above. Quasi concavity is guaranteed by concavity of U_i ($i=0,T$), while $W_1 > 0$ is implied by positive marginal utilities of current consumption and future wealth. This formulation allows for an additional motive for trade apart from information—a particular desire for or aversion to current consumption. This was used in Glosten and Milgrom (1985).

Example--Dynamic portfolio adjustment

Define $W(c, q; \omega)$ by:

$$W(c, q; \omega) = E[U(Y_T + \phi + c - R_2(q_2) - \dots - R_{T-1}(q_{T-1}) + (\nu + q + q_2 + \dots + q_{T-1})X_T | S)],$$

where q_i are the future optimal trades in the security, and R_i are the future terms of trade. If the investor's expectations of future terms of trade are independent of past trades, then concavity of U will imply quasi-concavity of W . If this independence does not hold, then it is not clear. In particular, some expectations over future terms of trade and some utility functions may invite "destabilizing trade" (a sequence of small buys followed by a large sale, for example). In this case, quasi-concavity is unlikely to hold for all ω . The possibility of future trades will be discussed further below.

Assumption 1 does rule out one specification that enjoys frequent academic consideration. That the marginal "utility" of cash is positive precludes the "pure noise trader" specification of Kyle (1985). While the general model admits a reasonably wide range of motives for trade, it still requires that investors care about the amount they pay for purchases or receive for sales.

The assumption of quasi-concavity means that characterization of an investor's decision is conveniently derived. The institution requires that the marginal price function be non-decreasing and that $R(q)$ be continuous.

Suppose for the moment that $R'(q)$ is continuous and defined for all q . Then the first order condition for maximization at D is:

$$-W_1(-R(D), D; \omega)R'(D) + W_2(-R(D), D; \omega) = 0; \text{ i.e., } R'(D) = W_2(-R(D), D; \omega)/W_1(-R(D), D; \omega).$$

If $R'(\cdot)$ is differentiable at D , then $R''(D)$ (the derivative of $R'(\cdot)$), is non-negative and the second order condition for local maximization at D holds since (suppressing the arguments of W , evaluating at D such that $R'(D) = W_2/W_1$ and recalling that $W_1 > 0$)

$$W_1^2 W_{22} + W_2^2 W_{11} - 2W_1 W_2 W_{12} - R''(D)W_1^3 \leq 0.$$

If $R'(q)$ is not differentiable at D , then the fact that $R'(\cdot)$ is non-decreasing will imply that the second order condition will hold for both the right and left hand limits.

The above analysis shows that if $R'(q)$ is continuous, any point satisfying the first order condition is a local maximum. Thus, if W is strictly quasi-concave, there is only one point satisfying the first order condition.

Now consider the case in which $R'(q)$ is discontinuous at a point Q , and define p_0 to be the limit from below of $R'(q)$ and p_1 to be the limit from above. The institution requires that $p_0 < p_1$. In this case, Q is a local maximum if:

$$p_0 \leq W_2(-R(Q), Q; \omega) / W_1(-R(Q), Q; \omega) \leq p_1.$$

Finally, if $R'(q)$ is defined for only an interval of q 's, say $[q_0, q_1]$, then q_0 will be a local optimum if:

$$R'(q_0) \geq W_2(-R(q_0), q_0; \omega) / W_1(-R(q_0), q_0; \omega);$$

while q_1 is a local optimum if:

$$R'(q_1) \leq W_2(-R(q_1), q_1; \omega) / W_1(-R(q_1), q_1; \omega).$$

Define the marginal valuation function at q and R of an investor with vector of characteristics ω to be $M(q, R; \omega) = W_2(-R, q; \omega) / W_1(-R, q; \omega)$. The above discussion relates local optima to the relation between the marginal valuation function at q and $R(q)$ with $R'(q)$. Strict quasi-concavity will imply that any local maximum is a global maximum. Therefore, there can be only one solution to the first order condition. If the marginal price is non-decreasing, then the marginal valuation function can cross the marginal price function at most once.

Lemma 1.

Suppose that W is strictly quasi-concave and that $R'(q)$ is any arbitrary non-decreasing marginal price function defined for q in the interval $[q_0, q_1]$ (q_0 may be negative infinity and q_1 may be

positive infinity). Then exactly one of the following mutually independent conditions holds:

- i. $M(q, R(q); \omega) > R'(q)$ for all q in $[q_0, q_1]$;
- ii. $M(q, R(q); \omega) < R'(q)$ for all q in $(q_0, q_1]$;
- iii. There exists exactly one $q^*(\omega) \in [q_0, q_1]$ such that:

$$q < q^*(\omega) \text{ implies } M(q, R(q); \omega) > R'(q)$$

$$q > q^*(\omega) \text{ implies } M(q, R(q); \omega) < R'(q).$$

Proof.

Suppose that i and ii do not hold, and suppose there exists $q(\omega)$ such that:

$M(q(\omega), R(q(\omega)); \omega) = R'(q(\omega))$. The derivative of $M(q, R(q); \omega)$ evaluated at $q(\omega)$ is:

$$(W_1^2 W_{22} + W_2^2 W_{11} - 2W_1 W_2 W_{12}) / W_1^3 < 0, \text{ by strict quasi-concavity.}$$

Thus, since $R'(\cdot)$ is non-decreasing, if M and R' ever meet, M crosses from above and hence conclusion iii follows. If there is no solution, $q(\omega)$, then either condition i or ii is satisfied, or there is a discontinuity in $R'(q)$ and M passes through this discontinuity. Since $R'(q)$ is non-decreasing, any discontinuity must involve a jump up. If M goes through this discontinuity it must do so from above and conclusion iii holds. Q. E. D.

The lemma illustrates the reason for assuming quasi-concavity. The optimal trade of an investor can be characterized as the solution to a first order condition. Strict quasi-concavity will make this solution unique. The characterization is provided in the discussion above the lemma, but for completeness, the results are collected in the following proposition.

Proposition 1.

Suppose that W is strictly quasi-concave for all ω and $R'(\cdot)$ is non-decreasing and defined for $q \in [q_0, q_1]$. Then an investor with vector of characteristics ω will choose $D_R(\omega)$ as the trade where

$D(\omega)$ is the unique solution to the following:

- i. if a solution to $M(q, R(q); \omega) = R'(q)$, then $D_R(\omega)$ is this unique solution
- ii. if the solution to the equation in i does not exist, but there is a point of discontinuity in R' at q^* and $M(q^*, R(q^*); \omega)$ lies between the limit from below q^* and the limit from above q^* , of $R'(q)$. then $D_R(\omega)$ is q^* .
- iii. if neither i nor ii hold, then $D_R(\omega) = q_1$ if $M(q, R(q); \omega) \geq R'(q)$ for all q and $D_R(\omega) = q_0$ if $M(q, R(q); \omega) \leq R'(q)$ for all q .

Proof.

Proof is immediate from the above lemma.

Before leaving the analysis of the individual investor, a corollary is provided that will be useful for the analysis in the subsequent sub-section. To the extent that investors have private information, limit order submitters may care about how individual investors value a share of the security. The following corollary shows the link between how investors value the security and the decisions that they make.

Corollary 1.

If W is strictly quasi-concave, and $R'(\cdot)$ is any non-decreasing marginal price function, then the following two sets of characteristic vectors are equivalent for any q :

$$\{\omega: D_R(\omega) \geq q\} = \{\omega: M(q, R(q); \omega) \geq R'(q)\};$$

$$\{\omega: D_R(\omega) \leq q\} = \{\omega: M(q, R(q); \omega) \leq R'(q)\};$$

Where $D_R(\omega)$ is defined in proposition 1 above.

Proof.

Proof is immediate from Lemma 1.

There may be marginal price functions that do decrease in some interval that also satisfy the conclusions of the corollary. That is, $R'(\cdot)$ being non-decreasing is not necessary for the conclusions. Any marginal price function that does satisfy the conclusions of the corollary shall be said to have the "single crossing" property. This will be important in the analysis of competing exchanges and market breakdown. What the property does is unambiguously link marginal valuations and trades.

II.B Equilibrium Bids and Offers

The sub-section above characterizes the behavior of investors taking the schedule of bids and offers as given. It is assumed that suppliers of liquidity—those who provide limit orders—recognize this behavior and take account of it in the provision of bids and offers. As stated in the introduction, a major part of this analysis focuses on the effects of asymmetric information. Rather than taking a particular parametric specification of information and division of information among potential investors, the assumption that defines the presence of private information will allow for a number of possible specifications.

The assumption to be made is in the spirit of the affiliation assumption in the auction literature (see for example Milgrom and Weber (1982)). In the case at hand, however, any quantity may be chosen, and hence the simple and elegant affiliation assumption of Milgrom and Weber is insufficient. To motivate the assumption, recognize that asymmetric information will be important to liquidity providers only if asymmetric information affects what individuals do. What individuals do is determined by their marginal valuation function and the terms of trade offered. Furthermore, the anonymity of the electronic market implies that all liquidity suppliers will observe about an arriving investor is that investor's marginal valuation at the trade chosen. This suggests that if there is private information that is of concern to liquidity suppliers, observing this point on the marginal valuation function must be informative. However, the private information assumption should not specify how

expectations will change in response to observing this point, since what point will be observed is a function of the bids and offers that prevail.

Recall that the marginal valuation at a trade Q involving cash flow R of a trader with characteristic vector ω is given by $M(q,R;\omega)$. From the point of view of the rest of the market, since ω is unobservable, the next arrival's marginal valuation function evaluated at Q and R is a random variable. The following assumption on the joint distribution of $M(q,R;\omega)$ and X , the future payoff on the security, is offered. It will be assumed that limit order submitters are risk neutral, and hence a condition on conditional expected values is all that is needed.

Assumption A2.

For each q and R and m , define the functions $V(m,q,R)$ and $v(m,q,R)$ by:

$$V(m,q,R) = E[X | M(q,R;\omega) \geq m]; \quad v(m,q,R) = E[X | M(q,R;\omega) \leq m]. \quad \text{The functions } V(.,.,.)$$

and $v(.,.,.)$ satisfy:

$$V(m,q,R) \geq E[X | M(q,R;\omega)=m] \geq v(m,q,R).$$

The economy exhibits strict adverse selection if the inequalities above are strict.

Since ω is never observed, were someone to observe the next arrival's marginal valuation evaluated at q and R , that individual would not know why the arrival had the particular marginal valuation. A high marginal valuation (given R and q) could be due to the investor being short in the security, it could be due to a relative aversion to current consumption, or it could be due to the investor having another source of income negatively correlated with the payoff X . What assumption A2 states is that one possibility for a high marginal valuation is the receipt of a signal indicating that the future payoff on the X is more likely to be large in the future. It should be noted that the inequality must hold for each q and R , and hence it is not an assumption about endogenous objects.

The assumption is implied by the condition that the "point" conditional expectation, $E[X|M(q,R;\omega)=m]$, be increasing in m . This follows since the upper and lower tail conditional expectations are weighted averages of point conditional expectations. The assumption is equivalent to the assumption that the functions $V(m,\dots)$ and $v(m,\dots)$ are both increasing in m . This and another useful property of these functions is proven in the following Lemma.

Lemma 2.

Assuming strict concavity of the investors' objective functions, and given assumption A2, the functions $V(m,q,R) = E[X|M(q,R;\omega) \geq m]$ and $v(m,q,R) = E[X|M(q,R;\omega) \leq m]$ are increasing in m , while $V(m,q,R+qm)$ and $v(m,q,R+qm)$ are increasing in q for all R and m .

Proof.

First note that if Y is a random variable with density f and distribution function F :

$$E[X | Y \geq y](1 - F(y)) = \int_y^{\infty} E[X | Y = t] f(t) dt$$

Similarly:

$$E[X | Y \leq y]F(y) = \int_{-\infty}^y E[X | Y = t] f(t) dt$$

Taking the derivatives of the above with respect to y shows that:

$$(d/dy)E[X|Y \geq y] = f(y)\{E[X|Y \geq y] - E[X|Y=y]\}/(1-F(y)),$$

$(d/dy)E[X|Y \leq y] = f(y)\{E[X|Y=y] - E[X|Y \leq y]\}/F(y)$. Given assumption A2 with $Y = M(q,R;\omega)$ shows that $V(m,q,R)$ and $v(m,q,R)$ are increasing in m .

For the second part of the proposition, define $Q_{Rm}(\omega)$ as the optimal trade of an investor with

characteristic vector ω but with cash position reduced by R facing a fixed price m for any quantity.

Such a "schedule" is non-decreasing, and hence by lemma 1 above:

$$E[X | Q_{Rm}(\omega) \geq q] = E[X | M(q, R + qm; \omega) \geq m] = V(m, q, R + qm)$$

$$E[X | Q_{Rm}(\omega) \leq q] = E[X | M(q, R + qm; \omega) \leq m] = v(m, q, R + qm).$$

Also, $E[X | Q_{Rm}(\omega) = q] = E[X | M(q, R + qm; \omega) = m]$. Thus, we have by assumption A2:

$$E[X | Q_{Rm}(\omega) \geq q] \geq E[X | Q_{Rm}(\omega) = q] \geq E[X | Q_{Rm}(\omega) \leq q].$$
 By the demonstration above, both

$E[X | Q_{Rm}(\omega) \geq q]$ and $E[X | Q_{Rm}(\omega) \leq q]$ are increasing in q . That is, both $V(m, q, R + qm)$ and

$v(m, q, R + qm)$ are increasing in q . Q. E. D.

The first result follows immediately from the observation that the expectation conditional on the marginal valuation being greater than or equal to m is an average of expectations conditional on the marginal valuation being equal to m' for $m' \geq m$. To understand the second part, note that $V(m, q, R + qm)$ is an average of expectations conditional on an investor choosing q or larger in an environment with a single ask price m . By the strict concavity of the investors' objective functions, an investor who chooses q or larger must have a marginal valuation at q that is m or larger. Thus the expectation conditional on an investor choosing q or larger exceeds the expectation conditional on an investor choosing q . The result follows.

The following provides some examples.

Example 1

Consider the environment of Glosten (1989). In that case, the next arrival has an endowment w , which, from the point of view of limit order submitters is normally distributed with mean zero. The future payoff of the security is X which is normally distributed. The next arrival has seen a signal $S = X + e$, with e normally distributed with mean zero, independent of X . Finally, the next

arrival maximizes the expected utility of future wealth, and the utility function is exponential with risk aversion parameter r . Standard calculations show that the marginal valuation is given by:

$$M(q, R; \omega) = E[X|S] - r\omega \text{VAR}(X|S) - r q \text{VAR}(X|S).$$

This example will be referred to below (call it the exponential, normal example), and it is convenient to choose some normalizations to minimize the number of parameters. If we interpret all conditional expectations and prices as deviations from the ex ante mean, we can choose the mean of X to be zero. Set $r\text{VAR}(X|S) = 1$, $\text{VAR}(\omega) = \alpha$ and $\text{VAR}(E[X|S]) = 1 - \alpha$. Roughly speaking, α is the proportion of the variance of trade explained by the liquidity motive. Then,

$$M(q, R; \omega) = \omega - q,$$

where $\omega = E[X|S] - r\text{VAR}(X|S)\omega$, and under the above assumptions, ω is a standard normal random variable. Furthermore, X and ω are correlated and $E[X|\omega] = (1 - \alpha)\omega$. Thus, the following holds:

$E[X|M(q, R; \omega) = m] = (1 - \alpha)(m + q)$. If $\alpha < 1$, this is strictly increasing in m and hence the assumption is satisfied.

Example 2

This example shows that the assumption is not innocuous. What can happen is that extreme marginal valuations could only come from uninformed investors. Suppose that there are informed agents and uninformed agents. Let U be a (zero, one) random variable which takes the value one if the next arrival is uninformed, and put $E[U] = \alpha$. Suppose that the uninformed have a marginal valuation given by $(\epsilon - q)$. Informed have seen the realization of some signal correlated with X , and they are risk neutral. Assume that U , ϵ , $E[X|S]$ are mutually independent, and $E[X] = 0$. Let $f(\cdot)$ denote the density of $E[X|S]$ and let $g(\cdot)$ be the density of ϵ . Then,

$$M(q, R; \omega) = (1 - U)E[X|S] + U(\epsilon - q), \text{ and } \omega = (U, S, \epsilon). \text{ Furthermore:}$$

$$E[X|M(q, R; \omega) = m] = (1 - \alpha)f(m)m / [(1 - \alpha)f(m) + \alpha g(m + q)]$$

While increasing for small m , this conditional expectation need not be increasing for all m and q . For example, suppose that f and g are both uniform densities, but the support of f is strictly contained in the support of g . Then, for extreme m , and some q the conditional expectation above will be zero and the assumption will not hold for all m and q . Call this the uniform example.

Note that the above two examples entail marginal valuations that were independent of the amount paid or received for a trade of q . This was, of course, due to the constant absolute risk aversion and the absence of wealth effects in the marginal valuation. The following provides an example which includes wealth effects. Suppose that X can be either 1 or 0. The signal S provides information on the likelihood that X will turn out to be 1. Let $p(s)$ denote $P\{X = 1 | S\}$. Suppose that the investor has (known) cash c and unknown endowment of shares, n . He or she is an expected utility of wealth maximizer and has log utility. The marginal valuation is given by:

$$M(q, R; \omega) = p(s)(c-R)/[c-R+(n+q)(1-p(s))].$$

It is required that $c-R > 0$ and $c-R+n+q > 0$. Note that the marginal valuation lies between zero and one. Suppose that n is uniformly distributed on $[0, A]$. Then, after some tedious calculations using Bayes' rule, one finds:

$$E[X | M(q, R; \omega) = m] = E[p(s)^2 / (1-p(s)) I_{\{L(m) \leq p(s) \leq U(m)\}}] / E[p(s) / (1-p(s)) / p(s) I_{\{L(m) \leq p(s) \leq U(m)\}}]$$

where $L(m) = m(q+c-R)/[qm+c-R]$, $U(m) = m(A+q+c-R)/[qm+qA+c-R]$.

This conditional expectation is increasing in m since $L(\cdot)$ and $U(\cdot)$ are increasing in m .

To derive the equilibrium among competing suppliers of liquidity (limit order submitters), the following assumption is made.

Assumption A3

Let N be the number of potential limit order submitters. Assume that N is large. Each limit

order submitter is risk neutral and has only publicly available information. Each liquidity supplier can provide any number of bids and offers. A limit order can be for any positive quantity.

In the immediately following analysis, it will be assumed that the set of prices at which limit orders can appear is a discrete set. Let this set of allowable prices be $P = \{...p_{-1}, p_0, p_1, ...\}$ where this set is arranged in increasing order. Let p_0 be the allowable price closest to the ex ante mean of X . That is, $p_{-1} < E[X] < p_1$. It seems reasonable, and will be proven below, that no risk neutral liquidity supplier will offer quantities at p_{-1} or below or bid for quantities at p_1 or higher. Given this set up, the strategy for each liquidity supplier consists of a specification of $\{q_i^A, q_i^B\} \geq 0$ where q_i^A is the quantity offered at price i and q_i^B is the quantity bid at price i . Quantities of zero are to be interpreted as no bid or offer provided. The analysis seeks the Nash equilibrium of the game in which liquidity suppliers expect investors to behave as derived in the subsection above. Each liquidity supplier observes the bids and offers of all other liquidity suppliers and chooses his or her optimal response.

The following analysis will deal with the derivation of the equilibrium on the offer side. The analysis for the bid side can be easily derived from this analysis. The following notation is used. Consider the problem of one of the liquidity suppliers, and let q_i be the quantity offered at the i th price. Let Q_i be the total quantity offered by all N liquidity suppliers at the i th price, and let AQ_i be the total quantity offered by all N liquidity suppliers at the i th price and lower. Finally, define R_i by $R_i = p_0 Q_0 + ... + p_i Q_i$, the amount paid for a purchase of AQ_i . Since the set of allowable prices is discrete, the marginal price function will be a step function. Thus, even if cross-sectionally the marginal valuation functions are continuously distributed, the probability that D , the quantity traded at the next arrival, is equal to AQ_i may be positive. In particular:

$$P\{D = AQ_i\} = P\{p_i \leq M(AQ_i, R_i; \omega) \leq p_{i+1}\}.$$

Denote the density of D , for $AQ_{i-1} < d < AQ_i$ by $f_i(d)$. Note that the above probabilities and densities are functions of the actual bids and offers provided.

If a trade arrives strictly between AQ_{i-1} and AQ_i , then the excess over AQ_{i-1} needs to be allocated among those supplying offers. It is assumed that such allocation is pro rata according to the size of offer provided. Since there is no time dimension to the provision of offers in this model, it is impossible to incorporate time priority. With this specification, the expected profit to the liquidity supplier who offers $\{q_i\}$ while others offer $\{Q_i - q_i\}$ is:

$$\sum_{i=0}^{\infty} q_i (p_i - E[X|D \geq AQ_i]) P\{D \geq AQ_i\} + \sum_{i=0}^{\infty} \left[\frac{q_i}{Q_i} \right] \int_{AQ_{i-1}}^{AQ_i} (p_i - E[X|D=d]) f_i(d) \quad (1)$$

This expression can be understood in the following way. If a liquidity supplier offers q_i , then all of this quantity will be transacted at price p_i if a trade comes in for AQ_i or greater. If this happens, the revised value of the share is $E[X|D \geq AQ_i]$ and this happens with probability $P\{D \geq AQ_i\}$. If a trade comes in for an amount strictly between AQ_{i-1} and AQ_i , say d , it will be allocated in a pro-rata fashion. The revised expectation will be $E[X|D=d]$. Integrating over all such d 's weighted by the density provides the expected profit in this event. Sum over all possible prices to obtain the expected profit from the choice of q 's.

To obtain the first order condition that Q_i must satisfy, take the derivative of the above expression with respect to q_i . This yields:

$$\begin{aligned}
& (p_i - E[X | D \geq AQ_i])P\{D \geq AQ_i\} + \frac{Q_i - q_i}{Q_i^2} \int_{AQ_{i-1}}^{AQ_i} (p_i - E[X | D=d])f_i(d) + \\
& + \sum_{j=1}^{\infty} \{q_j \frac{d}{dq_i} [(p_j - E[X | D \geq AQ_j])P\{D \geq AQ_j\}] + \frac{q_j}{Q_j} \frac{d}{dq_i} \int_{AQ_{j-1}}^{AQ_j} (p_j - E[X | D=d])f_j(d)\}
\end{aligned} \tag{2}$$

Sum this derivative over all liquidity suppliers and divide by N . This produces the following condition:

If $Q_i > 0$ but finite:

$$(p_i - E[X | D \geq AQ_i])p\{D \geq AQ_i\} + \left[\frac{(N-1)}{N}\right] \left[\frac{1}{Q_i}\right] \int_{AQ_{i-1}}^{AQ_i} (p_i - E[X | D=d])(d - AQ_{i-1})f_i(d) + K/N = 0. \tag{3}$$

The term K/N indicates a number of individual terms reflecting the effect of adding a unit of quantity more at p_i on the probability of trades larger than AQ_{i-1} . For large N we can ignore these terms, and the first order condition indicates that the expected profit from providing a unit of quantity is on the order of $1/N$. As N gets large, the first order condition becomes a zero profit condition.

After integrating the second term in (3) by parts, substituting $V(p_i, d, R_{i-1} + p_i(d - AQ_{i-1}))$ for $E[X | D \geq d]$ and ignoring terms of order $1/N$ it is found that if $Q_i > 0$, but finite:

$$\int_{AQ_{i-1}}^{AQ_i} (p_i - V(p_i, d, R_{i-1} + p_i(d - AQ_{i-1})))P\{M(d, R_{i-1} + p_i(d - AQ_{i-1})) \geq p_i\} = 0 \tag{4}$$

By Lemma 2, if $p_k < V(p_k, 0, 0)$, then $p_k < V(p_k, q, qp_k)$ for all positive q , and the first order

condition can never be satisfied at p_k . The equilibrium can thus be described as follows.

Define A_1 by $A_1 = \min\{p \in P: p > V(p,0,0)\}$ if the set is non-empty. If it is empty, then the book will provide no offers and the market closes down. If the set is non-empty, then A_1 is the lowest ask price. If the integral in the first order condition, (4) at price $p_i = A_1$ is positive for all q , then the quantity offered at A_1 will be infinite and description of the equilibrium is complete. If the first order condition is satisfied for finite q , then put Q_1^* equal to the solution. Now define A_2 by $A_2 = \min\{p \in P: p > V(p, Q_1^*, A_1, Q_1^*)\}$ if this set is non-empty. If it is empty, then the book offers only a quantity Q_1^* at a price A_1 . If it is non-empty, then A_2 is the second lowest ask. Determination of Q_2^* proceeds as above. Figure 1 displays the method of solution. Proposition 3 summarizes the equilibrium, and supplies the analogous results for the bid side.

Proposition 2.

Given the maintained assumptions, the following describes the equilibrium offers:

i. If $p < V(p,0,0)$ for all $p \in P$, then no offers are provided.

If $p > v(p,0,0)$ for all $p \in P$, then no bids are provided.

ii. If there exists a $p \in P$ satisfying $p > V(p,0,0)$, then the lowest ask, A_1 is the smallest such p .

If there exists a $p \in P$ satisfying $p < v(p,0,0)$, then the highest bid, B_1 is the largest such p .

iii. If the expression for the ask side first order condition, (4), with $p_i = A_1$ is positive for all q , then an infinite quantity will be offered at A_1 . Otherwise, the quantity offered at A_1 will be the solution to the first order condition. If the expression for the bid side first order condition with $p_i = B_1$ is positive for all q , then an infinite quantity will be bid at B_1 . Otherwise the quantity offered at B_1 will be the solution to the first order condition.

iv. If positive quantities are offered at k different ask prices, and putting AQ_k^* equal to the aggregate quantity offered at the k ask prices and putting R_k equal to the amount paid for the quantity AQ_k^* then:

- a. If $\{p \in P: p > V(p, AQ_k^*, R_k)\}$ is empty, then there are no higher offers.
- b. otherwise, put $A_{k+1} = \min\{p \in P: p > V(p, AQ_k^*, R_k)\}$
- c. If the integral in (4) with $p_j = A_{k+1}$ is non-negative for all q , then an infinite quantity is offered at A_{k+1} .
- d. otherwise, put Q_{k+1}^* equal to the solution to the first order condition.

If positive quantities are bid at k different bid prices, and putting BQ_k^* equal to the aggregate quantity bid at the k bid prices and putting R_k equal to the amount received for the quantity BQ_k^* then:

- a. if $\{p \in P: p < v(p, -BQ_k^*, -R_k)\}$ is empty, then there are not lower bids.
- b. Otherwise, put $B_{k+1} = \max\{p \in P: p < v(p, -BQ_k^*, -R_k)\}$
- c. if the first order condition with $p_j = B_{k+1}$ is non-negative for all q , then an infinite quantity is bid at B_{k+1} .
- d. Otherwise, put Q_{k+1}^* equal to the solution to the bid side first order condition.

Proof.

The only step of the proof left out in the above discussion is a verification that the first order condition satisfies the second order condition. The second order condition at a price p_i with a positive quantity is found by taking the derivative of the initial first order condition, (2), summing across all liquidity suppliers, dividing by N and ignoring terms of order $1/N$. This yields:

$P\{D_j \geq AQ_j\}(p_j - V(p_j, AQ_j, R_j))/Q_j < 0$ for the ask side and $P\{D \leq -BQ_j\}(p_j - v(p_j, -BQ_j, -R_j))/Q_j > 0$ for the bid side. The results of Lemma 2 imply that if the first order condition is

satisfied, so will the second order condition.

Some fairly general characteristics of the equilibrium fall out of the above derivation.

Consideration of these general characteristics give some insight into the driving forces of the equilibrium. They are collected in Proposition 3. Proposition 3 also provides the analogous results for the bid side of the book.

Proposition 3

Assume that $V(m,q,R)$ is strictly increasing in m , while $E[X|M(q,pq;\omega)=p]$ is continuous in q . Then,

i. If the market is open, then for ϵ small but positive,

$$A_1 > V(A_1, 0, 0) > v(B_1, 0, 0) > B_1; \text{ and } A_1 > E[X|M(\epsilon, \epsilon A_1; \omega) = A_1];$$

$$B_1 < E[X|M(-\epsilon, -\epsilon B_1; \omega) = B_1].$$

ii. if there are offers at k different ask prices, and bids at k different bid prices, then for ϵ positive but small:

$$E[X|D = AQ_{k-1} + \epsilon] < E[X|D \geq AQ_{k-1}^*] < A_k < E[X|D \geq AQ_k^*];$$

$$E[X|D = -BQ_{k-1} - \epsilon] > E[X|D \leq -BQ_{k-1}^*] > B_k > E[X|D \leq -BQ_k^*];$$

Proof.

The first inequality in i. follows immediately from the definition of A_1 . The second inequality follows from Assumption A2 and the third follows from the analogous definition of B_1 . The second set of inequalities follow from Assumption A2 and continuity. The same arguments apply for part ii.

Part i of the proposition shows that if the economy exhibits strict adverse selection, then the limit order book will have a positive bid ask spread no matter what the set of allowable prices is. That is, P can be made arbitrarily fine, and the small trade bid ask spread will persist. The reason for this is the possible trading on private information. An individual that provides an offer at the smallest ask price, will transact on every trade. Not only will he or she get a portion of small trades, but on all large trades, the total quantity offered will be taken. This means that in order to place an offer at the smallest ask, the individual has to be concerned with the informational implications of all investor purchases. The first part of the proposition also shows that small investor purchases and sales are profitable. Similarly, an individual placing a limit order at the largest bid needs to be concerned with the informational implications of all investor sales. The second part of the proposition stresses the importance of the "upper tail" expectations for the determination of offers and the "lower tail" expectations for the determination of bids. The proposition also shows that if the realized trade is just greater than AQ_{k-1} , that an offer at A_k will be profitable.

Part i of the proposition has a further implication. If the equilibrium does not provide an infinite quantity at any ask price, then every offer has a zero expected profit. But this implies that on average, averaged across all trades, the average price must equal the revised expectation. Since small trades are profitable, some larger trades must be unprofitable. That is, for small trades the average price paid by an investor exceeds the revised expectation, while for some larger trades the revised expectation is greater than the average price paid by an arriving investor.

Part ii of the proposition points out an interesting feature of the market. Suppose an order for $AQ_{k-1} + \epsilon$ arrives. This will clear out all the offers at A_1 through A_{k-1} , and part of the orders at A_k . The revised expectation in response to this realized trade lies strictly between B_1 and the now lowest ask price at A_k . Thus, there are no offers lying exposed below the revised expectation and no bids lying exposed above the revised expectation. It is not necessarily the case that offers need to be

canceled after this trade. Even though the model assumes constant vigilance on the part of limit order submitters, constant monitoring need not be necessary to avoid unfavorable trades.

The import of the proposition and the above observations are illustrated in Figure 2. Define $e(q)$ by $e(q) = E[X|D=q] = E[X|M(q,R(q)\omega)=R'(q)]$ where $R'(q)$ is the equilibrium marginal price function (a step function) and $R(q)$ is its Lebesgue integral.

Examples

Before proceeding to a further analysis of the electronic open limit order book, it is perhaps informative to examine some examples of the above general analysis. First consider the normal, exponential example introduced above. Recall that $E[X|M(q,R;\omega) = m] = (1-\alpha)(m+q)$. Thus, if f is the standard normal density and F the standard normal distribution function, $V(m,q,R)$ is given by:

$$V(m,q,R) = (1-\alpha)f(m+q)/(1-F(m+q)).$$

As long as α is positive, there exists a solution to $p = V(p,q,R)$ for all q . Thus, the order book will, in principle, provide terms of trade for arbitrarily large orders. In fact, if the set of prices is coarse enough, and α is large enough, an infinite quantity may be offered at A_1 . This will happen if:

$A_1(f(A_1)/(1-F(A_1)) - A_1) \geq 1-\alpha$. This condition is found by evaluating (4) at $p_1=A_1$ and Q_1 equal to plus infinity and finding the condition under which the integral is non-negative. This can happen if the price set is very discrete and α is large.

The second example provides a somewhat different equilibrium. Recall the uniform example discussed above:

$$M(q,R;\omega) = (1-U)E[X|S] + U(e-q),$$

where U , $E[X|S]$, and e are mutually independent, $E[U] = \alpha$, and suppose that $E[X|S]$ and e are both uniformly distributed on $[-L,L]$. In this case, for $L > M > 0$, $q \geq 0$:

$V(m,q,R) = (1-\alpha)(L^2 - m^2)/[2(1-\alpha)(L-m) + 2\alpha(L-m-q)I_{\{q \leq L-m\}}]$, where I_E is the indicator function of the set E . In particular, $V(m,0,0) = (1-\alpha)(L+m)/2$. As long as the set of prices is not too coarse and/or α is large enough, some quantity will be offered. All that is required is that there be an allowable price in the interval $((1-\alpha)L/(1+\alpha), L)$. However, arbitrarily large trades will not be possible in this environment. Since, 1) at any ask an infinite quantity will not be offered and 2) if $q > L(1 - (1-\alpha^2)^{.5})/\alpha$ the function $V(m,q,R)$ lies above m for all m . Thus, after the book has provided a quantity up to the above limit or higher, no subsequent offers will arrive. The exact quantity to be provided will depend upon the allowable price set and the other parameters. However, that the quantity offered will be finite is true no matter what the allowable price set. It can also be verified that the example will work if e has a somewhat wider support than $E[X|S]$ despite the fact that the "affiliation" assumption, A2, is violated.

Finally, an example is provided in which the market will not open. Suppose that all investors are risk neutral, and some investors have information while others do not. Then, the marginal valuation of the next arrival is:

$$M(q,R;\omega) = (1-U)E[X|S] + UE[X]. \text{ Then, for } m > E[X],$$

$V(m,0,0) = E[X|(1-U)E[X|S] + UE[X] \geq m] = E[E[X|S]|E[X|S] \geq m] > m$. Of course, this example does not conform to the assumption of strict quasi-concavity of the objective function. An example which conforms to the strict quasi-concavity assumption could be constructed by considering a situation in which some investors are informed, while it is common knowledge that all investors have no other risky component of their portfolio, and start with a zero position in the security in question.

For the remainder of the analysis, it will be convenient to drop the assumption that only a discrete set of prices is allowed. While admittedly unrealistic, the mathematics is simplified tremendously. It should be noted that relatively few of the characteristics derived above in the general

analysis and the specific examples relied on the particular set of allowable prices. The passage to continuous prices will be accomplished by taking limits of the discrete analysis above as the set of prices becomes finer. Thus, one may think of the continuous price case as a mathematically convenient approximation to the more realistic step function marginal price schedule.

As above, the analysis will deal with the ask side; the analysis for the bid side is completely analogous. The limit as q goes to zero from above of $R'(q)$ is:

$R'_+(0) = \inf\{p: p > V(p,0,0)\}$ if the set is non empty. If empty, there are no offers provided. Now suppose that offers totaling Q are available. The following limiting argument will indicate the conditions that $R'(Q)$ and $R(Q)$ must satisfy. Suppose that $R'(Q)+e$ is the next allowable price, and further that a positive quantity will be offered at $R'(Q)+e$. Following the development above, this implies that:

$$R'(Q)+e > V(R'(Q)+e, Q, R(Q)).$$

Let the quantity offered at $R'(Q)+e$ be ϵq . Then the first order condition must be satisfied:

$$\frac{\int_Q^{Q+\epsilon q} [R'(Q)+e - V(R'(Q)+e, t, R(Q) + (t-Q)(R'(Q)+e))] P\{M(t, R(Q) + (t-Q)(R'(Q)+e)) \geq R'(Q)+e\} dt}{\epsilon q} = 0$$

Taking the limit as e goes to zero yields:

$$R'(Q) = V(R'(Q), Q, R(Q)).$$

It is also required that $1 > [V(R'(Q)+e, Q, R(Q)) - R'(Q)]/e$. Taking limits yields the additional condition that $(\partial/\partial p)V(p, Q, R(Q))|_{p=R'(Q)} \leq 1$. Thus, $R(Q)$ for $Q > 0$ is a solution to a differential equation with the initial condition $R(0)=0$. There may still be more than one solution. The second condition states that a solution be picked with $V_1(R'(Q), Q, R(Q)) \leq 1$. Finally, the solution is pinned down by the condition for $R'_+(0)$.

The differential equation condition states that the marginal price at a quantity Q must satisfy $R'(Q) = E[X|D \geq Q]$. That this should be so; i.e. the marginal price is determined by the "upper tail" expectation was discussed in the context of the discrete price equilibrium. Going to the continuous price limit merely changes set of inequalities into an equality. The second condition guarantees that the solution to the differential equation leads to an increasing marginal price function. Letting V_i denote the partial derivative of V with respect to its i th argument, differentiating the differential equation condition leads to:

$R''(q) = (V_2 + V_3 R'(q))/(1 - V_1)$. Proposition 2 showed that $V(R'(q), q + \delta, R(q) + \delta R'(q))$ is increasing in δ . Thus, the numerator is positive.

As already noted, a solution need not exist. However, an appeal to the limiting argument shows that if there are an interval of p 's such that $p > V(p, 0, 0)$ then a solution will exist for some interval of quantities. The above observations are collected in Proposition 4.

Proposition 4.

For $Q > 0$, the marginal price function $R'(Q)$ must satisfy:

$$R'(Q) = V(R'(Q), Q, R(Q)) = E[X|M(Q, R(Q)); \omega] \geq R'(Q)]$$

$V_1(R'(Q), Q, R(Q)) \leq 1$, $R(0) = 0$, $R_+'(0) = \inf\{p: p > V(p, 0, 0)\}$, where $R_+'(0)$ is the limit as q goes to zero from above of $R'(q)$.

For $Q < 0$, the marginal price function must satisfy:

$$R'(Q) = v(R'(Q), Q, R(Q)) = E[X|M(Q, R(Q)); \omega] \leq R'(Q)]$$

$v_1(R'(Q), Q, R(Q)) \leq 1$, $R(0) = 0$, $R_-'(0) = \sup\{p: p < v(p, 0, 0)\}$, where $R_-'(0)$ is the limit as q goes to zero from below of $R'(q)$.

A solution to this system will exist for some interval of quantities, Q if $m > V(m, 0, 0)$ for some interval of m 's and $m < v(m, 0, 0)$ for some interval of m 's.

As in the discrete market, $R_+'(0) > R_-'(0)$, and $\lim(q < 0) E[X|D=q] > R_-'(0)$ and

$\lim(q>0)E[X|D=q] < R_+'(0)$. A typical equilibrium is illustrated in Figure 3.

Examples

In the exponential, normal example, we have

$$R'(q) = (1-\alpha)f(R'(q)+q)/(1-F(R'(q)+q)) \text{ for } q > 0$$

$$R'(q) = -(1-\alpha)f(R'(q)+q)/F(R'(q)+q) \text{ for } q < 0.$$

The equilibrium can be illustrated in a neater form by deriving the equilibrium trade by an individual of type z . Denote $q(z)$ as the solution to $z-q(z) = R'(q(z))$. Then, for $q(z) > 0$:

$$z-q(z) = (1-\alpha)f(z)/(1-F(z)) \text{ as long as } z > z^*, \text{ the solution to } z^* - (1-\alpha)f(z^*)/(1-F(z^*)) = 0.$$

For $z < -z^*$, the solution is given by $q(z) = z + (1-\alpha)f(z)/F(z) < 0$. Notice that z^* is the limit as q goes to zero from above of $R'(q)$.

For the uniform distribution example, $R'(q)$ is the solution to a quadratic equation. Depending upon q , the quadratic equation has two roots, one root or no roots. If two roots are available, the partial derivative condition $V_1 \leq 1$ requires taking the smaller root. The lack of a root indicates that a marginal price is not offered for that quantity. Using the expression for V developed above in the previous discussion of the example, $R'(q)$ for $q > 0$ is:

$$R'(q) = \{L - \alpha q - (\alpha^2 q^2 - 2\alpha q L + \alpha^2 L^2)^{-5}\} / (1+\alpha) \text{ for } q < L(1-(1-\alpha^2)^{-5})/\alpha.$$

III. Further Characteristics of The Electronic Market

One characteristic of a trading mechanism that may be important is its ability to consistently provide some liquidity. The ability of the monopolist specialist system to provide liquidity was the focus of Glosten (1989). The key property that allowed a specialist to keep the market open when the competitive mechanism considered there closed down was the ability of the specialist to average profits across trades. Notice, that this is a feature that the electronic market considered here shares

with the specialist system. A reasonable question is whether this electronic market does it as well. The answer to this question is a restricted yes, in the sense that, in some economies, if the electronic market provides no liquidity (formally there is no finite solution for $R'(q)$), then any other market mechanism that has a "nice" marginal price function will expect to lose money. Thus, a large set of markets will be open in an environment only if the electronic exchange would be open in that environment.

Proposition 5

Suppose that there is no finite fixed point, $m, m = V(m,0,0)$ so that the electronic market will not open, and assume an economy in which marginal valuations are independent of cash positions so that $V(m,q,R)$ is independent of R . Then any other price schedule which has the single crossing property (see the discussion following Corollary 1) will expect to lose money.

Proof.

For any arbitrary revenue schedule $R(\cdot)$, the expected profits on investor purchases are:

$$\int_{-\infty}^{\infty} dP\{Q_R \leq q\} (R(q) - qE[X|Q_R = q])$$

Integrate by parts to get

$$\int_0^{\infty} P\{Q_R \geq q\} (R'(q) - E[X|Q_R \geq q]) dq + \int_{-\infty}^0 P\{Q_R \leq q\} (E[X|Q_R \leq q] - R'(q)) dq$$

This follows since

$$(d/dq)P\{Q_R \geq q\}E[X|Q_R \geq q] = -E[X|Q_R = q](d/dq)P\{Q_R \leq q\}, \text{ and}$$

$$(d/dq)P\{Q_R \leq q\}E[X|Q_R \leq q] = E[X|Q_R = q](d/dq)P\{Q_R \leq q\} \text{ (see the proof of Lemma 2).}$$

Under the hypothesis of the proposition, $m < V(m,0,0) \leq V(m,q,R(q))$ for all m . The second

inequality follows from Lemma 2 and the absence of wealth effects. Then in particular, for the $R'(q)$ considered here:

$R'(q) < V(R'(q), q, R(q)) = E[X | M(q, R(q); \omega) \geq R'(q)] = E[X | Q_R \geq q]$, where the last equality follows from the single crossing property. Thus, this R leads to negative expected profits.
Q. E. D.

For the electronic market to open, all that is required is that liquidity suppliers be willing to make a small trade. Any other exchange, if open, would have to make this small trade, plus trades that are worse from an informational perspective. Thus, if the liquidity suppliers are unwilling to provide quotes, others would be unwilling as well. The proposition leaves open the possibility that in an economy with important wealth effects, a schedule might be designed to capitalize on these wealth effects and remain open when the electronic exchange closes. Thus, for small q , the marginal offer will be less than the upper tail expectation, but possibly larger q 's would show the opposite relation. I have been unable to either verify this possibility or disprove the possibility.

One can measure liquidity in a variety of ways. Based on the size of the small trade spread, one might be tempted to say the electronic market is not liquid. Indeed, it is possible to specify an economic environment in which a non-discriminating (or single price) electronic market has no small trade spread. This is the example of competitive pricing in Glosten (1989). However, such a market might close down too quickly. The above proposition states that if the measure of liquidity is resilience in the face of severe adverse selection problems, then the electronic market as conceived here is as good as one can do.

On the other hand, if the electronic market is open for some quantities, then a monopolist specialist would keep the market open as well. The proposition raises the possibility that an electronic market may be able to reap the benefits of competition while at the same time preserving the

monopolist specialist liquidity in the face of severe adverse selection problems. I have no general answer to this, but the normal, exponential example that has been considered above indicates that in at least one environment, this statement is true. Proposition 6 provides the details.

Proposition 6.

Consider the normal, exponential example. All traders are no worse off, and many are strictly better off with the open limit order book than with a monopolist specialist.

Proof.

Under the normalization chosen above, the certainty equivalent of a trader of type ω , making optimal trade $q(\omega)$ is given by:

$CE(\omega) = \omega q(\omega) - .5q(\omega)^2 - R(q(\omega))$. The derivative of this is given by:

$CE'(\omega) = q(\omega) + q'(\omega)(\omega - q(\omega) - R'(q(\omega))) = q(\omega)$ since $q(\omega)$ satisfies the first order condition for optimality. Since the certainty equivalent is zero when the optimal quantity traded is zero, the certainty equivalent evaluated at ω is the integral from any ω^* such that $q(\omega^*) = 0$ to ω of $q(t)$. A monopolist will set a marginal price schedule so that the quantity traded by an investor of type w is given by:

$$\begin{aligned} q_m(\omega) &= \alpha\omega - (1-F(\omega))/f(\omega), \omega > \omega_m \\ &= \alpha\omega + F(\omega)/f(\omega), \omega < -\omega_m \end{aligned}$$

0 otherwise; where $\alpha\omega_m - (1-F(\omega_m))/f(\omega_m) = 0$. The details of this derivation are in Glosten (1989). In contrast, the electronic market determines $q_e(\omega)$ as:

$$\begin{aligned} q_e(\omega) &= \omega - (1-\alpha)f(\omega)/(1-F(\omega)), \omega > \omega^* \\ &= \omega + (1-\alpha)f(\omega)/F(\omega), \omega < -\omega^* \\ &0, \text{ otherwise.} \end{aligned}$$

It can be shown that $0 < f(t)[(f(t)/(1-F(t)))-t]/(1-F(t)) < 1$. Hence, for $\omega > \omega^*$, $q_e(\omega) > q_m(\omega)$ and

for $\omega < -\omega^*$, $q_e(\omega) < q_m(\omega)$. Thus, for ω outside of $[-\omega^*, \omega^*]$ the certainty equivalent is strictly larger with the electronic market. Q. E. D.

A market has been derived that breaks even on average, yet in general, average transaction prices will not equal revised expectations. Furthermore, since $R'(q)$ is increasing, the average price schedule is upward sloping. This might suggest that the investor has an incentive to break up trades into a sequence of smaller trades. However, there is a countervailing effect. The marginal price schedule is upward sloping because trade leads to revisions in expectations. Thus, a sequence of small trades will move the entire price schedule. While I have been unable to verify that there is no incentive to break up trades, it can be shown that an investor will not choose to break up a trade into a large number of very small trades if he or she believes the marginal price schedule will merely be moved up by the trade. The reason is that a very large number of small trades moves the schedule substantially. Thus, the incentive to split up trades, if it is there, is not as strong as might be supposed.

Proposition 7.

Let $C(n)$ be the cost of purchasing q/n shares n times. Suppose that the shape of the marginal price function remains unchanged over time (i.e., it merely moves up or down in response to changes in expectations) while $e(q) = E[X | M(q, R(q); \omega) = R'(q)] > E[X]$ and is bounded away from $E[X]$. Then as n gets large, $C(n)$ gets large.

Proof.

Each time a purchase of q/n is made, it costs $R(q/n)$, and it pushes up the expectation of the value of the stock by $e(q/n) - E[X]$. This increase must be paid in all future purchases. Thus, $C(n)$ is given by:

$$C(n) = R(q/n) + \{(q/n)(e(q/n) - E[X]) + R(q/n)\} + \{(q/n)^2(e(q/n) - E[X]) + R(q/n)\} + \dots \\ + \{(q/n)^{n-1}(e(q/n) - E[X]) + R(q/n)\}$$

$= (n-1)n(q/n)(e(q/n) - E[X])/2 + nR(q/n)$. As n gets large, the second term approaches $qR_+'(0)$, while the first term is on the order of:

$$q(n-1)(e(0) - E[X])/2.$$

By assumption of the proposition this becomes large as n gets large. Q. E. D.

It should be stressed that the purpose of the proposition is not so much to prove that there is no incentive to break up trades, but rather to show that the incentive may not be as large as first supposed. In fact, the hypotheses of the proposition are questionable. If there are informed traders, then on average, they will come in to sell in response to the price schedule being moved up by the sequence of small trades. These "corrections" will increase the incentive to break up a trade. On the other hand, the corrections are risky, and a risk averse investor may not choose to expose him or herself to future uncertainties about the terms of trade. Finally, if an investor did have expectations about future marginal price schedules as hypothesized, then this investor would believe the following strategy would be profitable: make a large number of very small buys and then reverse the position with one large sell. Dynamic equilibrium in this market should rule out investors believing this strategy would be profitable.

IV. Competition Among Exchanges

This section of the paper considers competition among exchanges, and asks how susceptible the electronic exchange and other conceivable exchanges are to entry of competitors. To do this analysis, the paper considers a wide open regulatory environment in which anyone can offer to make a market in the security. Furthermore, setting up such a "market" is costless. On the investor side,

market orders can be costlessly split up among "exchanges." It turns out that this assumption is a very powerful one, and is a driving force behind the results. This is put formally as Assumption A4.

Assumption A4.

In the presence of more than one exchange, an investor can costlessly and simultaneously send separate orders to each exchange. A competing exchange can be costlessly established and supplies a marginal price schedule which satisfies the single crossing property (see corollary 1).

The first question to be asked is whether, given the existence of the electronic exchange, any potential entrant would be willing to enter. The standard Nash assumption is made--the entrant takes the revenue function of the electronic market as given. It might appear that since small trades are profitable for the electronic market that there will be an incentive to offer a price schedule to capture these small trades and skim the cream. This will not work because if small orders find it profitable to go to the competing exchange, then all investors will find it profitable to send some part of their order to the competing exchange. Even were the quantity accepted by the competing market limited, it would still get a portion of all trades. The structure of the proof is as follows. Since investors optimally split their orders, the marginal price received will be the marginal price in the electronic exchange. This marginal price is the upper tail expectation if there were only the electronic market. However, this artificial upper tail expectation is less than the actual upper tail expectation if the quantity traded in the competing market is positive since upper tail expectations are increasing in quantity (in a world with no wealth effects). Thus, the competing market will consistently receive marginal prices that are less than the upper tail conditional expectations. However, expected profit is a weighted average of the marginal price less the upper tail conditional expectation.

Proposition 8.

Assume an economy in which marginal valuations are unaffected by cash positions so that $V(m, q, R)$ is independent of R . Suppose $R_e'(q)$ satisfying $R_e'(q) = V(R_e'(q), q, R(q))$ is the marginal price schedule in the electronic exchange. Assuming this price schedule fixed, an entrant with a marginal price schedule satisfying the single crossing property will expect to make no money.

Proof.

Call $R_c'(q)$ the marginal price schedule in the competing market, and Q_c , a random variable, the next trade at the competing exchange. After integrating by parts, the expected profit to the entrant is:

$$\int_0^{\infty} P\{Q_c \geq q\} (R_c'(q) - E[X | Q_c \geq q]) dq + \int_{-\infty}^0 P\{Q_c \leq q\} (E[X | Q_c \leq q] - R_c'(q)) dq$$

Where Q_c is the quantity chosen in the entering market. Consider only the offer side. If $R_c'(0) > R_e'(0)$ then if $Q_c > 0$, $Q_e > 0$, where Q_e is the quantity chosen in the electronic market.

Furthermore, $R_c'(Q_c) = R_e'(Q_e)$ and hence $Q_e = R_e'^{-1}(R_c'(Q_c))$. To simplify the notation, define $q_e = R_e'^{-1}(R_c'(q))$, $q_T = q_e + q$ and $R_T = R_c(q) + R_e(q_e)$. That is, q_e is the trade made in the electronic market when q is traded in the competitive market, q_T is the total trade, while R_T is the total amount paid for a purchase of q_T shares. By the single crossing property, the events

$\{Q_c \geq q\} = \{Q_e \geq q_e\}$, and furthermore:

$$\begin{aligned} E[X | Q_c \geq q] &= E[X | Q_e \geq q_e] \\ &= E[X | M(q_T, R_T; \omega) \geq R_c'(q)] \\ &= V(R_c'(q), q_T, R_T) = V(R_e'(q_e), q_T, R_T) \\ &\geq V(R_e'(q_e), q_e, R_e(q_e)) = R_e'(q_e) = R_c'(q). \end{aligned}$$

The last inequality follows from the fact that $q_T > q_e$ and the use of Lemma 2 in the case of no

wealth effects.

Thus, for any q such that $R_c'(q) \geq R_c'(0)$ the term in the integral is non-positive. Suppose that

$R_c'(q) < R_c'(0)$. Then,

$$\begin{aligned} R_c'(q) - E[X|Q \geq q] &= R_c'(q) - E[X|M(q, R_c(q); \omega) \geq R_c'(q)] \\ &= R_c'(q) - V(R_c'(q), q, R_c(q)). \end{aligned}$$

Since $R_c'(q) < R_c'(0)$ and $V(R_c'(q), q, R_c(q)) \geq V(R_c'(q), 0, 0)$ this term is not positive since $R_c'(0)$ is the smallest m with $m \geq V(m, 0, 0)$.

The proposition asserts that, in a sense, the electronic market is competition proof. One Nash equilibrium is that there will be no entrance. The proposition is almost, but not quite, trivial. After all, an entrant supplying a competing non-decreasing schedule could as easily provide this schedule by participating in the limit order book. The assertion of equilibrium in the limit order book implies that there are no profit opportunities and that any such effort would lead to negative profits. The slight addition is the allowance of marginal price schedules to have some downward sloping portion as long as the single crossing property is satisfied. What the proposition provides is the first hint that the competition in the discriminatory limit order book mimics the competition among exchanges. This point will, it is hoped, become clearer with subsequent results.

Reference to the proof above suggests, that should an entrant come in, unless the limit orders change, limit order submitters will lose money as well. Thus, there may be another equilibrium in which there is entrance. In fact, one can be fairly sure there will be other equilibria. For example, two competing open limit order books, each offering half the liquidity provided by a single limit order book will be an equilibrium. The result will be terms of trade identical to those provided by a single order book. The next proposition shows that this is more generally true: if the entrant makes non-negative profits, the price schedule by the two markets replicates the price schedule that would be

determined if there were only the electronic exchange. The proof uses the same approach as above. If there are two exchanges, the marginal price received in the competing exchange will be driven by the marginal price in the open limit order book. But this is determined to be an upper tail conditional expectation taking account of the existence of the other exchange. Thus, in every case, as long as the competing exchange does not undercut for small trades, the marginal price equals the upper tail expectation. But this is precisely the equilibrium when there is only one order book. The non-negative profit assumption rules out undercutting at small trades.

Proposition 9.

Suppose that there is an equilibrium in which a competing market enters and supplies a marginal price schedule $R_e'(q)$, satisfying the single crossing property. Then there is an equilibrium in which the total revenue function $R_T(q) = R_e(q_e) + R_e(q_e) (q_e + q_e = q)$ is equal to $R(q)$ the schedule determined when there is only the electronic market.

Proof.

If both q_e and q_e are positive for some q , then they are determined by: $q_e + q_e = q$ and $R_e'(q_e) = R_e'(q_e)$. Thus, $R_T'(q)$ is given as $R_e'(q_e)$. Then:

$$\begin{aligned} R_T'(q) &= R_e'(q_e) = E[X | M(q, R_T(q); \omega) \geq R_e'(q_e)] \\ &= V(R_e'(q_e), q_T, R_T(q)) = V(R_T'(q), q, R_T(q)). \end{aligned}$$

That is, $R_T(q)$ is a solution to $R_T'(q) = V(R_T'(q), q, R_T(q))$. One such solution is the electronic open limit order book solution, $R(q)$. The entrant cannot set $R_e'(0) < R_e'(0)$ and expect to make non-negative profits, for if he or she did, some marginal prices would be below upper tail expectations while other marginal prices would equal upper tail expectations. Q. E. D.

The above two propositions state that if there is a great deal of competition in the provision of

limit orders, any additional competition is either unprofitable or redundant. The question that remains to be answered, however, is whether this result is due merely to the great deal of competition that has been assumed, or does the actual architecture of the discriminatory limit order book play a role. The next proposition shows that the architecture is important. It is the particular zero profit condition determined by the architecture of the discriminating limit order book that discourages further competition. Specifically, any other exchange that expects non-negative profits but does not replicate the electronic exchange will invite entrants.

Proposition 10.

Consider an exchange with marginal price function $R'(q)$, and suppose that for some interval of q 's it does not equal the electronic exchange marginal price schedule. Suppose further that this schedule has non-negative expected trading profits and satisfies the single crossing property. Then, holding this schedule constant, there exists a competing schedule that will earn positive profits.

Proof.

Suppose without loss of generality that the schedule diverges from the electronic exchange schedule on the offer side. If

$$\int_0^{\infty} P\{Q \geq q\} (R'(q) - E[X|Q \geq q]) \geq 0$$

but $R'(q)$ is not the electronic exchange marginal price schedule, then there exists q^* with $R'(q^*) > E[X|Q \geq q^*]$. Consider the following strategy of an entrant. Set $P = R'(q^*)$ and announce that up to Q units will be sold at price P . The expected profit from this strategy is:

$$\int_0^Q dP\{Q_c < q\} q (P - E[X|Q_c = q]) + Q P\{Q_c = Q\}(P - E[X|Q_c = Q])$$

where Q_c is the random quantity picked in the competing market. From the investors maximization problem $\{Q_c = Q\} = \{M(Q + q^*, R(q^*) + PQ; \omega) \geq P\}$, and hence $E[X|Q_c = Q] = V(P, Q + q^*, R(q^*) + PQ)$. Divide the expression for profits by Q and let Q go to zero. The first term vanishes, the second becomes

$$(P - V(P, q^*, R(q^*)))P\{M(q^*, R(q^*); \omega) \geq P\} = (R'(q^*) - V(R'(q^*), q^*, R(q^*)))P\{M(q^*, R(q^*); \omega) \geq R'(q^*)\} > 0.$$

Thus, for some $Q > 0$ expected trading profits will be positive. Q. E. D.

The idea of the proof is that if an entrant offers a small quantity, every investor with marginal valuation greater than or equal to the price offered will be interested in trading with the entrant. Thus, the cost of supplying the offer is the conditional upper tail expectation. By hypothesis, the price is greater than the upper tail expectation and the entrant expects to make money. The proof of this proposition shows that while the electronic exchange is not open to cream skimming (see for example, Glosten (1991)), any other exchange is. The proposition implies that the particular design of the electronic market is important. That is, it is not just the competition among a large number of liquidity suppliers that leads to the resilience of the electronic exchange. For example, an alternative design of an electronic market would be a "non-discriminating" exchange. Liquidity suppliers submit limit bids and offers for quantities of the security. If a market order to purchase q units arrives, then the first limit orders totaling q all transact at the price of the highest offer to transact. Equilibrium among the large number of liquidity suppliers dictates that the price for an order of size q , $P(q)$ satisfy $P(q) = E[X|Q=q]$. In this case, $R(q)$ is given by $P(q)q$ and hence we have $R(q)/q = E[X|M(q, R(q); \omega) = R'(q)]$.

In the event that there is no private information, both designs will yield the same result--all

bids and offers will stack up at $E[X]$. If there is private information, however, the two designs will lead to different revenue functions. Recall that with private information the original specification of the electronic market had $R_+'(0) > E[X]$. Taking limits of the above expression for the alternative design "non-discriminating" exchange, $R'(0) = E[X|M(0,0)=R'(0)]$. In some environments (for example, the normal, exponential example) the solution to this is $R'(0) = E[X]$. Thus, the alternative design will have $R'(q) < E[X|Q \geq q]$ for q small. Since the exchange will earn zero profits on average, for larger q the opposite inequality must hold. The above proposition demonstrates that such an exchange will invite competition.

It should be added that the analysis in Glosten (1989) shows that the non-discriminatory exchange will break down if the adverse selection problem is too severe. Thus, the analysis has suggested two reasons for preferring a discriminating design: it is less likely to break down and does not invite competitive reaction. The comparison is not unambiguous, however, since the non-discriminating form will tend to offer lower spreads for small quantities.

IV. Extensions and Speculations

Perhaps the most arbitrary of the assumptions made in this analysis is the strict dichotomy between those who supply limit orders and those who trade against the book. While it might be reasonable to assume that those with information not use limit orders, it is probably true that some "liquidity" traders would use limit orders. This is particularly true if access to the book were very inexpensive. If the model were to allow this, it is possible to make rough predictions about the results. Consider the discrete price analysis. There is now no longer any reason to expect a zero profit condition to hold at every price where there is positive quantities. A liquidity trader may be willing to experience negative expected trading profits in return for more optimally balanced portfolio

and consumption. However, if there are positive profits at some price, one would expect the patient traders to step in to remove those profits. This would suggest that the resulting marginal price function would offer larger aggregate quantities at each price than the schedule considered here.

Black (1992) considers an environment in which only risk neutral informed trade against the book while limit orders are supplied by liquidity traders. In this case, the marginal valuation of a trader is $E[X|S]$. Thus, $R'(q) = E[X|D=q] = E[X|E[X|S]=R'(q)]$, and hence $R(q)/q < E[X|Q=q]$. Limit order submitters may be willing to take the loss because of the liquidity motive for trade. The schedule determined will depend upon how much loss the liquidity traders are willing to take. Black hypothesizes that were the model presented here to allow (or perhaps require) liquidity traders to use limit orders and if a "no destabilizing trade" assumption were invoked, that the equilibrium would look identical to his. This could be, but I believe that the notion of "no destabilizing trade" is a difficult one to rigorously specify. While requiring that the marginal price schedule be linear and rise twice as fast as the average price may be sufficient, it may not be necessary. Further analysis of dynamics is required.

The assumption of a large number of "patient traders" providing limit orders is unlikely to be met in reality. After all, providing limit orders is in fact not costless since it requires some monitoring to insure that orders are not left exposed after, for example, a public information release. As the discussion of the discrete price case suggests, the quantity competition that results in this sort of environment does not lead to the "Bertrand" conclusion that $N=2$ is large. Of course, if there are a small number of liquidity suppliers, then there is an incentive for others to provide terms of trade. It is probably cheapest, however, for such liquidity suppliers to merely join the book by providing limit orders and thus compete directly with the "patient traders."

As the uniform example illustrates, it is not difficult to come up with reasonable examples that do not conform to the "affiliation" assumption, A2. A failure of this assumption to hold may mean

that the resulting pattern of bids and offers is roughly upward sloping, but involve many "flat" spots-- prices at which a large quantity is bid or offered.

Several of the propositions regarding the comparison of the open limit order book with other exchanges limited the comparison to those exchanges providing "nicely behaved" marginal price schedules. In particular, marginal price schedules for which the investor's first order condition had a unique solution were considered. My guess is that a deeper analysis would show that this limitation is not necessary (given the remainder of the assumptions). It would seem that all that should matter is the behavior of the investor's marginal valuation function in the neighborhood of the solution chosen. By quasi-concavity of the investor's objective function, at any solution the marginal valuation function is downward sloping, and crosses the marginal price from above. Thus, the set of types who will chose a quantity just above q does correspond to those whose marginal valuations are above $R'(q)$. The method of analysis in the propositions seems to require the more global statement embodied in the single crossing property, and I have been unable to relax this requirement.

I am not nearly so sanguine about the robustness of the results to consideration of non-anonymous exchanges. I believe it possible and indeed likely that exchange floors may provide the sort of information that allows either (1) some further determination of who does and does not have information or (2) the possibility of disciplining via future penalties, those who make information based trades. Indeed, Admati and Pfleiderer (1991) argue that (1) can occur via "sunshine trading." Benveniste and Wilhelm (1991) argue that (2) is an important role of the specialist and floor traders. Specialists themselves will insist that these other sources of information are important for the smooth running of the NYSE. It is likely, that this floor information is important for some trades, unimportant for others. I believe that an important area of research is to first determine the importance of this other information and second to determine if the securities industry can simultaneously enjoy the benefits of competition and liquidity that an open limit order book appears to

provide with the information benefits that a floor may provide.

VI Conclusion

After setting up a reasonably general model of investor behavior, the paper develops some characteristics of the equilibrium in an electronic market when there are a large number of limit order submitters. It is shown that the equilibrium involves an "upper (lower) tail" conditional expectation in the determination of offers (bids). While exhibiting a small trade spread, the open limit order book provides as much liquidity as can be expected in extreme adverse selection environments. Despite the fact that the marginal price schedule is upward sloping in the equilibrium, there is not a strong incentive to break orders up into many small orders. The paper suggests that if there is a large population of potential liquidity suppliers, and if the actual costs of running an exchange are small, then among exchanges that operate continuously and anonymously, and supply nice marginal price schedules, the electronic exchange is the only one that does not tend to engender additional competing exchanges. The analysis ignores two possibly important comparisons. How does a continuously operating open limit order book fair against periodic call markets? How does an open limit order book fair against non-anonymous exchanges. The first analysis requires not only a reasonable model of why people trade, but a model of why people trade when they do. The second requires a further analyses in the spirit of Admati and Pfleiderer (1991) and Benveniste and Wilhelm (1991) as well as an empirical understanding of the importance of floor information.

Perhaps the most useful way to interpret the results regarding competing exchanges is as follows. With an electronic open limit order book, a competing exchange may well survive. The analysis suggests that if it is to survive it must provide something outside of the analysis in this paper. Additional trading information is a likely candidate for that something, and hence the paper does not predict the demise of exchange floors. It does suggest, however, that organized exchanges (with a

reputation for accurate clearing) might seriously consider offering an electronic open limit order book in addition to their floor trading services.

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Figure 1A

Graphical illustration of the determination of the first two ask prices, $A_1 = p_i$ and $A_2 = p_j$.

$V(m,0,0) = E[X | M(0,0,\omega) \geq m]$, $V(m,Q_1^*,A_1Q_1^*) = E[X | M(Q_1^*,A_1Q_1^*;\omega) \geq m]$.

$A_1 = \min\{p \in P : p > V(p,0,0)\}$, $A_2 = \min\{p \in P : p > V(p,Q_1^*,A_1Q_1^*)\}$. Q_1^* satisfies the condition in (4).

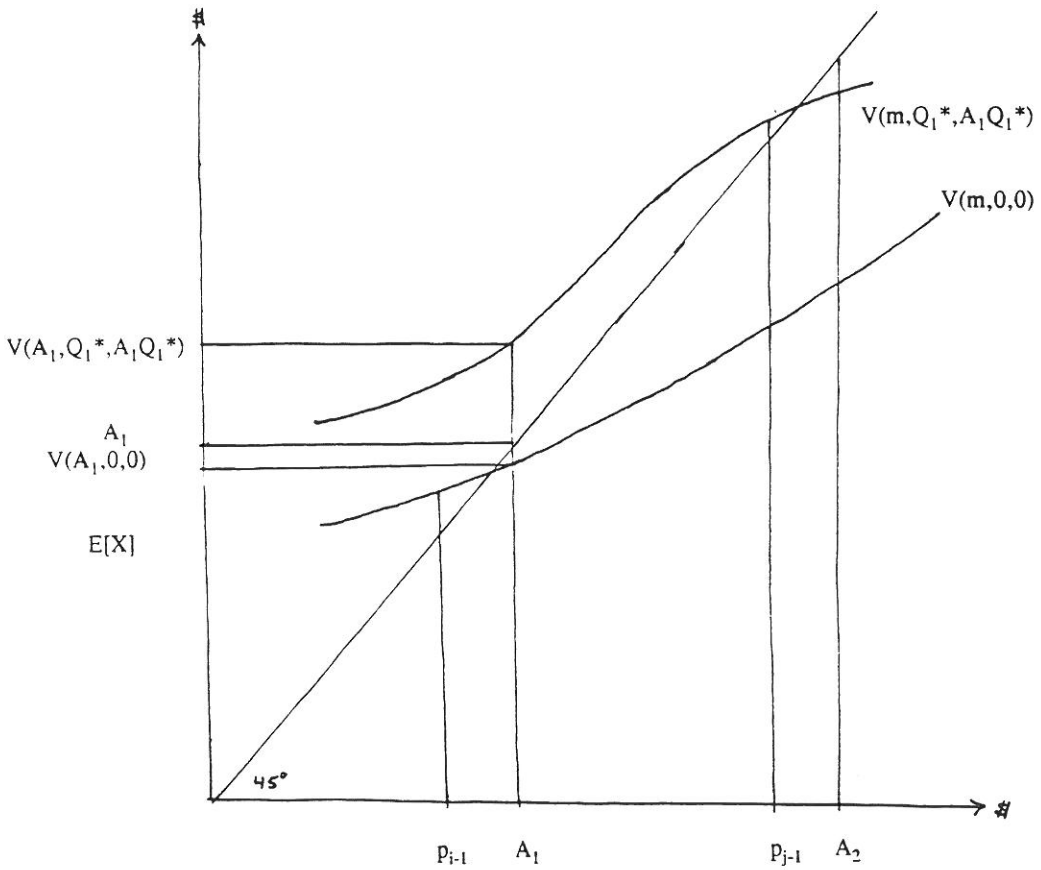


Figure 1B

Illustration of the case in which only Q_1^* is offered at $A_1 = p_i$.

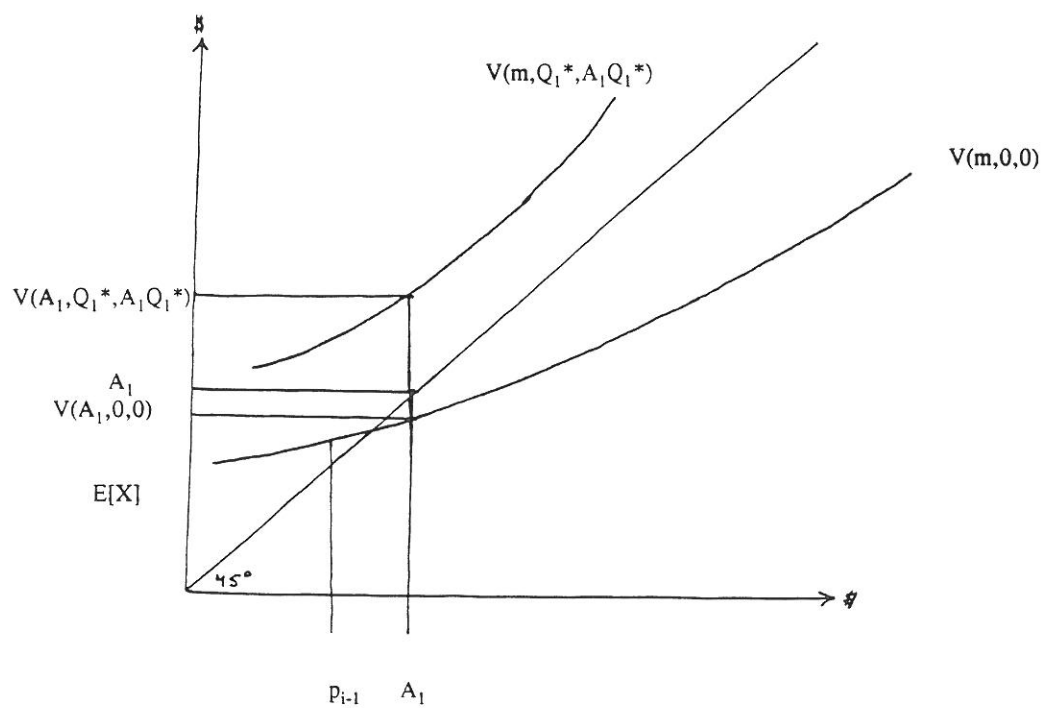


Figure 1C

Illustration of the case in which the electronic market fails to open

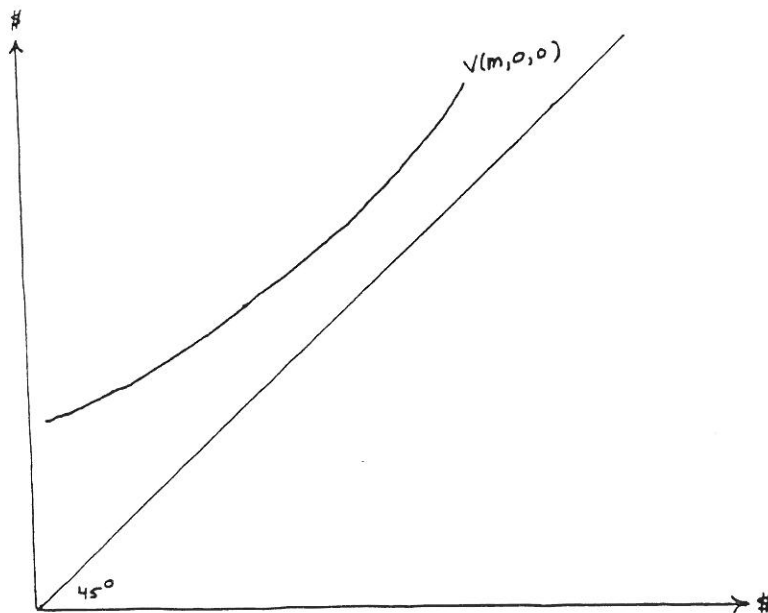


Figure 2

Illustration of the equilibrium marginal price schedule, $R'(q)$, and the revision in expectations function, $e(q) = E[X|M(q, R(q); \omega) = q]$ and the average price schedule, $R(q)/q$.

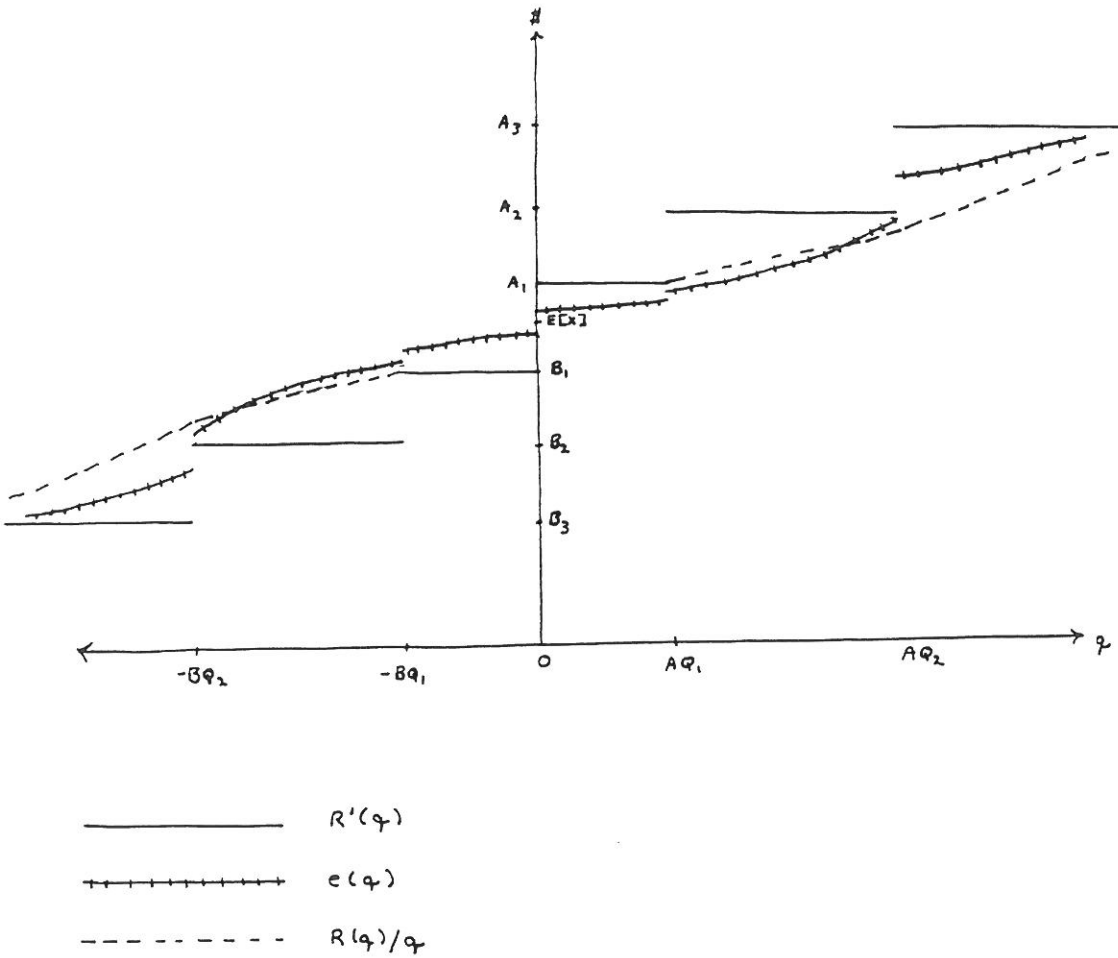
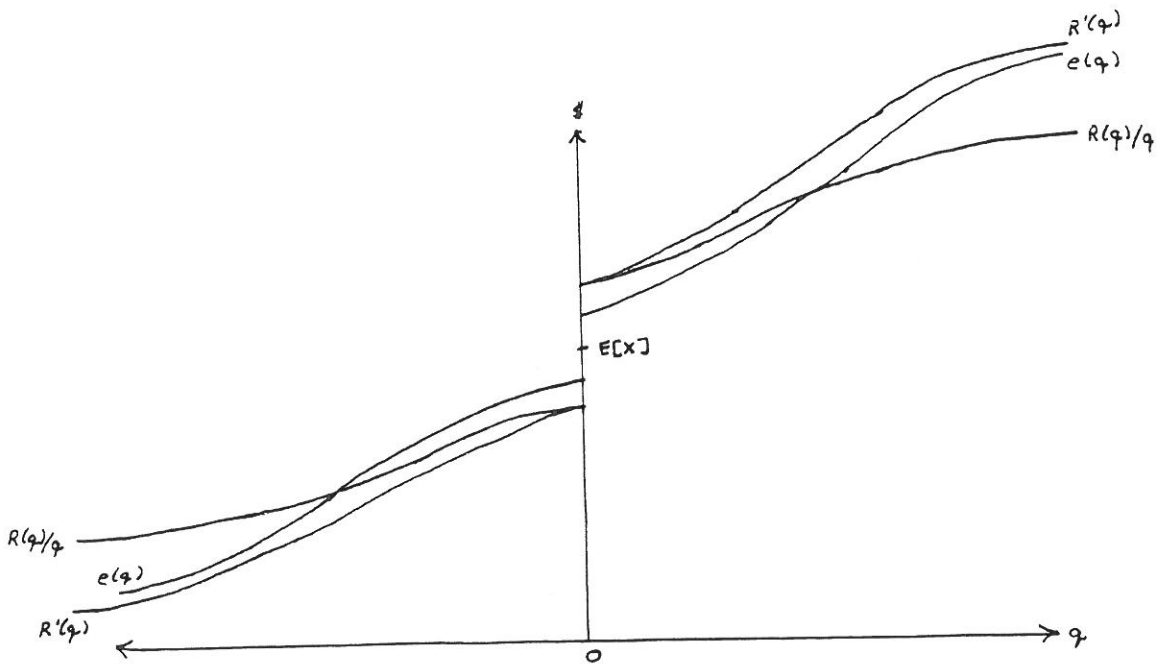


Figure 3

Illustration of the equilibrium marginal price schedule, $R'(q)$, the revision in expectation function $e(q) = E[X|M(q, R(q); \omega) = R'(q)]$ and the average price schedule $R(q)/q$.



Jämvikt i en elektronisk orderbok med öppen limit

Denna uppsats härleder ett schema för jämviktspriser bestämda av sälj- och köpkurser i en orderbok med öppen limit. Detta sker under relativt generella antaganden. Analysen visar att orderboken får en liten skillnad mellan köp- och säljkurser och att små transaktioner blir lönsamma, medan större inte blir det. Vidare framkommer att en elektronisk börs erbjuder största möjliga likviditet i extrema situationer. En analys av konkurrerande börser som är anonyma ger vid handen att den elektroniska börsen inte inbjuder till konkurrens, eftersom en nystartad börs skulle förränta sig icke-positiva handelsvinster. Skulle en konkurrerande börs upprättas i alla fall skulle den intjäna icke-negativ vinst endast ifall det prisschema som möter placeraren såg likadant ut som det schema som upprättats på den elektroniska börsen. Den elektroniska börsen är entydigt (inom klassen av anonyma börser) immun mot konkurrens. Den allmänna analysen belyses med exempel.

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